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PARTIAL ORDERINGS AND MOORE-SMITH LIMITS*

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1. Introduction. Everybody knows that the concept of limit is fundamental in analysis. But somewhere about the time he reaches advanced calculus, a student may perhaps begin to wonder how many disguises this concept of limit can assume. Apart from the fact that the dependent variable which is doing the converging may be a real number or a complex number or a vector or a function or what not else, the independent variable may also take any one of a multitude of forms. Thus if the dependent variable is real, we still have to consider convergence of sequences of reals, of multiple sequences, of functions of a real variable x as x tends to some x_0 or as x tends to infinity, of functions of several real variables, and so on. The saving feature is of course that all these assorted definitions have strong resemblances. Nevertheless, the student may be forgiven if he wishes that somebody would put all these different but related ideas into fewer packages, so that it wouldn't be necessary to do almost the same thing over and over again for the slightly different kinds of limit. This may occur to him even if he has never heard of E. H. Moore's dictum, that the existence of strong similarities in the central part of different theories indicates the existence of a more general theory of which these different theories are all special cases. But in such a situation as this there is something to be considered beside generality. What we really want is a treatment of the subject which is not only unified, but is also elegant and easily understood. As a matter of fact, such a theory has been in existence for twenty-nine years. The treatment of limits† devised by E. H. Moore and his then student H. L. Smith appeals to me as one which can readily fit into a course in the theory of functions of a real variable or a course in advanced calculus, not only strengthening the content but also providing a continuous thread tying the subject matter together. And besides this, their treatment has another virtue which is important to a beginner and by no means to be despised by those who are no longer beginners; it follows closely the lines of the simplest of all limit theories, that is the theory of convergent sequences, follows them indeed so closely that many of the proofs can be taken over with only trivial notational changes. Thus the theorems for the more complicated limiting processes are obtained along with the simpler ones, and at a cost in effort little greater than that demanded by the simplest of all.

Let us then look over some of the definitions of limit that an undergraduate might be expected to know, and try to separate the essential parts shared by the different examples. From one point of view, we could classify all the defini-

* Presented as an invited address before the Mathematical Association of America Dec. 30, 1950.

† Moore, E. H. and Smith, H. L., A general theory of limits. Amer. J. Math., vol. 44, 1922, p. 102. Here we should also mention the very inclusive theory of limits based on filters, devised by H. Cartan, Theorie des filtres, C. R. Acad. Sci. Paris, vol. 205, 1937, pp. 595–598, Filtres et ultrafiltres, ibid., pp. 777–779. This same theory was also expounded by H. L. Smith, apparently independently (A general theory of limits, Nat. Math. Mag., vol. 12, 1938, pp. 371–379).

tions into two principal varieties. The first type is exemplified by the limit of a function $f(x)$, defined for all real x, as x approaches x_0. Roughly stated, the limit of f is l if $f(x)$ is near l whenever x is near x_0. The other type is exemplified by the limit of a sequence. The limit of f_n $(n=1, 2, \cdots)$ as "n tends to infinity" is l if f_n is near l for all n after a certain n_0. If we would generalize the first type, we should study the idea "x is near x_0" and seek to give it a meaning both precise and general. Thus we are led to study neighborhoods, and eventually arrive at the idea of a topological space. In this day it would be utterly superfluous to stress the importance of the study of topological spaces. But at the moment we are not going in that direction. It is the other type of limit that is to engage our attention.

2. Directed sets and nets. What then are the essential elements of the definition of the limit of a sequence? To begin with, the property that n is an integer is quite irrelevant since the same kind of definition applies to the limit of $f(x)$ as x tends to ∞ over the whole real number system. But the relation "n is after n_0," or in symbols $n>n_0$, is important. To begin with, it is a binary relation on the integers. That is, if we write an integer n, and then write the name $>$ of the relation, and then write another integer m, we obtain a meaningful sentence "$n>m$"; the subject is "n," the verb is "$>$," and the object is "m." The sentence may be true or it may be false, but it is not nonsense. This is what we shall mean by a binary relation on a set A; R is a binary relation on A if whenever a and b are members of A, "aRb" is a meaningful sentence, either true or false.

But a relation that could in any reasonable sense be thought of as "after" must have another property. If a is after b and b is after c, we surely can reasonably ask that it should also be true that a is after c. This is the property called "transitivity." Formally, a binary relation R on A is transitive if whenever a, b and c are elements of A such that aRb and bRc, then it is true that aRc. We shall consider that binary relations R with this property are important enough to be dignified by the name of "partial orderings"; so henceforth a "partial ordering" of A shall be a transitive binary relation on A. (But not all mathematicians agree on this meaning, or on any other, for the expression "partial ordering.") Also, we shall make a slight notational change as a crutch to memory; instead of the non-committal name R for the partial ordering, we shall usually use $\succ$, as a reminder of its kinship with $>$. The statement "$a\succ b$" could be read "a follows b."

However, we have not yet completed our task of finding all the essential properties used in the definition of a sequence. To show this, we observe that if we define $n\succ m$ to mean that $n-m$ is a positive even integer, $\succ$ is a partial ordering. We could repeat the definition of limit of a sequence with the single change of replacing $>$ by $\succ$, and obtain an intelligible definition. But if we define f_n to be 0 if n is even and 1 if n is odd, we find that $f_n=0$ if $n\succ 2$, so $\lim f_n$ is 0; while $f_n=1$ if $n\succ 3$, so $\lim f_n$ is 1. This is most undesirable. Let us

therefore look back at the proof that a sequence cannot have two different limits. Suppose that h and $k \neq h$ are both limits of f_n; let ϵ be a positive number less than the difference between h and k. For all n after a certain n' we have f_n within $\epsilon/2$ of h; for all n after a certain n'' we have f_n within $\epsilon/2$ of k. *Now choose an n which is after both n' and n''.* For this n, f_n differs by less than $\epsilon/2$ from both h and k, which therefore must be within ϵ of each other. But this contradicts the definition of ϵ, and so the sequence cannot have two different limits. If we try to carry this through with the relation $\succ$ in place of $>$, we find that the trouble occurs at the italicized sentence; if n' is even and n'' is odd, there is no n such that $n \succ n'$ and $n \succ n''$. This partial ordering has the peculiarity that if $n \succ m$, then if m is even so is n, and if m is odd so is n.

Accordingly, we avoid the trouble of non-unique limits by limiting our attention to those partial orderings in which this situation never arises. That is, we assume that our partial orderings have the property that for each pair of members of the set (not necessarily distinct) there is an element of the set which follows both of them. Moore and Smith gave this property a name* which has fallen into disuse, probably because the property has no importance except when combined with transitivity. When a relation $\succ$ partially orders a set A and also has the property just described, it is now customary to say that the relation $\succ$ "directs" the set A. To state it in full:

A set A is directed by a relation $\succ$ if $\succ$ is a binary relation on A with the properties:

(i) *if a, b, and c are elements of A such that $a \succ b$ and $b \succ c$, then $a \succ c$;*

(ii) *If a and b are elements of A, there exists an element c of A such that $c \succ a$ and $c \succ b$.*

One more bit of terminology is called for. A function which assigns to each positive integer n a real number f_n as functional value has long been known as a sequence of real numbers; more generally, a function which assigns to each n a functional value f_n in a set M is called a sequence of elements of M. We wish to have a name for a function which assigns to each element a of a directed set A a real number $f(a)$ as functional value. For this J. L. Kelley† proposes the name "net"‡ of real numbers; more generally, if f is a function which assigns to each element a of a directed set A a functional value $f(a)$ in a set M, we shall call the function f a "net" of elements of M.

3. Definition of convergence. Now we are in a position to attempt to take over the whole theory of convergent sequences, replacing the word "sequence"

* Compositive.

† J. L. Kelley, *Convergence in topology*, Duke Math. J., vol. 17, 1950, p. 277.

‡ Kelley writes me that this was suggested by Norman Steenrod in a conversation between Kelley, Steenrod and Paul Halmos. Kelley's own inclination was to the name "way"; the analogue of a subsequence would then be a "subway"! Since a stream with its tributaries is a good example of a system directed by the relation "downstream from," I incline toward "stream" rather than "net." But "net" has the great advantage of having seen print first.

by "net," the sign $>$ by $\succ$, and the integers by the elements of the directed set A. For example, the definition of limit becomes:

DEFINITION. *Let $f(a)$, a in A, be a net of real numbers, and let k be a real number. Then $\lim_{a \text{ in } A, \succ} f(a) = k$ means that for every positive ϵ there is an element a_ϵ of A such that $|f(a)-k| < \epsilon$ whenever $a \succ a_\epsilon$.*

For simplicity of notation we shall omit such parts of the symbolism under lim as can be left out without danger of confusion. For example, if $\succ$ is the only partial order of A that we are considering, we omit mentioning it; if all the functions we are interested in at a given moment are defined on the same directed set A, we condense the symbol to $\lim_a f(a)$ or even to $\lim f$. But this is a device familiar to all of us. By such simple devices we take over all the essential parts of the theory of limits. For example, if $f(a)$ and $g(a)$ are both defined and real valued for all a in a directed set A, and have the respective finite limits h and k, then $f+g$ has the limit $h+k$. As another important theorem, let f be a net of real numbers defined on a directed set A and having the property that $f(a) \geqq f(a')$ whenever $a \succ a'$. (Such nets may be called "monotone non-decreasing," but I think that Garrett Birkhoff's terminology "isotone" is much to be preferred. The corresponding name for nets such that $f(a) \leqq f(a')$ when ever $a \succ a'$ is "antitone," to replace the older expression "monotone non-increasing.") The ordinary proof for sequences carries over to show that if f has this property and also has a finite upper bound, it has a limit, and the limit is the same as the least upper bound.

The proof of the Cauchy criterion for convergence could be taken over too, but we shall sketch one which has the virtue of applying in any metric space by merely replacing absolute differences by distances. Suppose it known that the Cauchy condition is necessary and sufficient for the convergence of sequences. Let $f(a)$, a in A, be a net which satisfies the Cauchy condition for nets; that is, for each positive ϵ there is an a_ϵ in A such that $|f(a)-f(a')| < \epsilon$ whenever both a and a' are $\succ a_\epsilon$. We choose successively $a_1, a_2, \cdots$ in A such that $a_n \succ a_{n-1}$ and $|f(a)-f(a')| < 1/n$ whenever a and a' are $\succ a_n$. Then the numbers $f(a_n)$, $n=1, 2, \cdots$ form a Cauchy sequence, so they have a limit k. Now to show that the net has k as limit, let ϵ be a positive number, and pick $n > 2/\epsilon$, so that $|f(a_{n+1})-k| < \epsilon/2$. If $a \succ a_n$, by definition of a_n we have $|f(a_{n+1})-f(a)| < 1/n < \epsilon/2$, so $|f(a)-k| < \epsilon$, completing the proof. This particular theorem is quite convenient, since it provides at once all the assorted forms of the Cauchy criterion for all the various limit processes occurring in advanced calculus and theory of functions of a real variable.

There is, however, one portion of the general theory on which we ask deferment for a few moments. This is the idea of subsequence and the theorems connected with the idea. Before we take this up, we wish to stop to look at some special cases. For we have now generalized the theory of limits from the simple cases of ordinary sequences to something more general, and it is natural that we should pause to see how much new territory we have taken in.

4. Examples. In showing that some special kind of limit process is covered by the Moore-Smith theory there is a simple pattern that we often use. The symbol for the limit usually has some notation under the letters lim that indicates, pictorially speaking, in which direction the independent variable is "going." For instance, the limit of a sequence $\lim_{n\to\infty} f_n$ is a limit "as n goes to infinity." To change over to the notation of nets we try to put a partial ordering on the independent variable in such a way that $a > b$ shall means that a has "gone further" than b. Thus in the case of sequences, $m > n$ should mean that m has "gone further toward infinity than n has," that is $m > n$. In the case of a limit $\lim_{x\to c} f(x)$, $x > x'$ should mean that x has approached closer to c than x' has, that is $|x-c| < |x'-c|$. In this case, though, we add as usual the requirement that x and x' should not equal c.

As a first trial, we consider a real valued function f defined on a set A of real numbers having no finite upper bound, and look at $\lim_{x\to\infty} f(x)$, wherein we of course understand that x is restricted to A. According to the pattern described in the preceding paragraph, we define $x > x'$ to mean $x > x'$. It is trivially easy to show that A is directed by $>$, and that the Moore-Smith limit $\lim_{x \text{ in } A, >} f(x)$ is the same as $\lim_{x\to\infty} f(x)$. For a slightly less trivial example we take a double sequence, $f_{m,n}$ $(m, n = 1, 2, \cdots)$ of reals. Since we want the limit as both m and n tend to infinity, we define $(m, n) > (m', n')$ to mean that $m > m'$ and $n > n'$. We easily prove that the set of all pairs of positive integers is directed by this relation and that the Moore-Smith limit in this case reduces to $\lim_{m,n\to\infty} f_{m,n}$. The same device applies to sequences with more than two subscripts.

Suppose that A is a set of real numbers, and p a real number such that arbitrarily near it are numbers $a \neq p$ in A. Let f be a real valued function on A. We wish to show that $\lim_{x\to p} f(x)$ is a special case of the Moore-Smith limit. So we define $x > x'$ to mean that neither x nor x' is p and that $|x-p| < |x'-p|$. The Moore-Smith definition reduces to the usual $\epsilon-\delta$-definition except that in it the δ's are restricted to the form $|x-p|$, x in A. This restriction is easily seen to be without effect, so $\lim_{x\to p} f(x)$ is another example of the Moore-Smith theory. The same device can equally well be applied in spaces of any number of dimensions, and in fact in any metric space.

We shall now discuss unordered sums, historically the first step toward the Moore-Smith theory.* Let X be any set, and f a real-valued function on X. Even when X has an order-relation, for instance when X is the set of positive integers, we wish to form partial sums with more and more elements, chosen without regard to order. So we let A consist of all finite subsets of X, and for each a_1 and a_2 in A we define $a_1 > a_2$ to mean that a_1 contains a_2, that is all points belonging to a_2 belong to a_1. This set is clearly directed, since for any a and b in A the set c consisting of all points in a and all points in b satisfies $c > a$, $c > b$. If a is in A and consists say of $x_1, \cdots, x_k$, we define $S(a)$ to mean $f(x_1) + \cdots + f(x_k)$. This is a net; and $\lim S = k$ means that to each positive

* E. H. Moore, *Definition of limit in general analysis*, Proc. Nat. Acad. Sci., vol. 1, 1915, pp 628–632.

ϵ corresponds a finite set a_ϵ such that whenever a is a finite subset of A containing a_ϵ, consisting say of $x_1, \cdots, x_h$, then $|f(x_1)+\cdots+f(x_h)-k|<\epsilon$. It is not difficult to see that when X consists of the positive integers, lim S exists if and only if the series $f(1)+f(2)+\cdots$ is unconditionally convergent, in which case lim $S=\sum_{n=1}^{\infty} f(n)$.

Next we consider a still less traditional looking kind of net. Suppose that C is a plane curve defined by equations $x=x(t)$; $y=y(t)$, $a\leqq t\leqq b$. We wish to give precision to the idea that the length of C is the limit, in some sense, of the lengths of polygons inscribed in C as they acquire more and more vertices. So for A we choose the set of all polygons inscribed in C, the word "inscribed" being understood in the following sense. In $[a, b]$ we choose points $t_0=a<t_1<t_2<\cdots<t_{n-1}<t_n=b$, and we join the points $\{x(t_0), y(t_0)\}$, $\{x(t_1), y(t_1)\}$, $\cdots$, $\{x(t_n), y(t_n)\}$ in that order by line segments. The result is a polygon inscribed in C. If P and P' are in A, we say that $P>P'$ if and only if all the vertices of P' are also vertices of P. This is easily seen to satisfy (i) of the definition of direction; and if P' and P'' are in A, we can construct an inscribed polygon whose vertices are all the vertices of P' and all those of P'', so that $P>P'$ and $P>P''$, and part (ii) of the definition is also satisfied. Let $L(P)$ be the length of P. This is a net of real numbers. Its limit, if it exists, is called the length of the curve C. Since we easily see that $L(P')\geqq L(P'')$ whenever $P'>P''$, the general theorem on isotone (or non-decreasing) nets tells us that the limit is the same as the least upper bound, so the length of C is also the least upper bound of the lengths of all polygons inscribed in C.

5. Application to the definition of the integral. Our next example is more important, because it is one which is often unsatisfactorily treated in text-books. Let f be defined and bounded on an interval $[a, b]$ of real numbers; we wish to define its (Riemann) integral. First we choose numbers $a=x_0<x_1<\cdots<x_n=n$, and other numbers $\xi_1, \cdots, \xi_n$ such that ξ_i is between x_{i-1} and x_i. Then we form the sum $\sum_{i=1}^{n}(x_i-x_{i-1})f(\xi_i)$, and we take some kind of limit of this sum. The question is, though, what kind of limit. Sometimes one hears "the limit as the number of subdivisions tends to infinity," with some qualifying remark about the lengths of the subdivisions. But this sum is not a single-valued function of the number of subdivisions. Some kind of extension of the concept of limit is called for. One alternative is to look sternly at the student and say "That's perfectly clear, isn't it?" This is what Mark Kac calls "proof by intimidation." A better way is to widen the theory to cover multiple-valued functions, treating the value of the sum as a multiple-valued function of the length of the longest subinterval (x_{i-1}, x_i). A third way, which we now discuss, is to use the Moore-Smith limit. Let us use the name "partition" for a system of division points $a=x_0<x_1<\cdots<x_n=b$ together with the intermediate points $\xi_1, \cdots, \xi_n$ satisfying $x_{i-1}\leqq\xi_i\leqq x_i$; the "norm" of this partition shall be the greatest of the numbers $x_1-x_0, x_2-x_1, \cdots, x_n-x_{n-1}$. If P is the partition just described, we use it to determine a sum $S(P)=\sum_i f(\xi_i)(x_i-x_{i-1})$. This is a single-valued function

of the partition P; in fact, it was just for the purpose of making the sum single-valued that we defined P to consist of both division and intermediate points. Now we wish to express the integral as the limit of $S(P)$ as something happens. This something may be thought of in either of two ways. We may say that we want the longest of the subintervals in the partition to approach zero, or we may say that we want the limit as we cut up the interval (a, b) finer and finer. If we choose the first of these points of view, we would say that for two partitions P and P', the statement $P \succ P'$ should mean that the norm of P is less than the norm of P'. If we choose the other point of view, we would say that the statement $P \succ P'$ means that all the points of division $x_0', x_1', \cdots, x_m'$ occurring in P' are among the points $x_0, x_1, \cdots, x_n$ occurring in P. In either case, the intermediate points ξ_i are disregarded in defining the order. For the kind of integral we are here discussing it does not matter which of the two definitions of order we choose; if the finite sum has a limit when one of the two definitions of $\succ$ is used it has the same limit when the other is used, and this limit is by definition the (Riemann) integral of f. The choice of limiting process is in this case a matter of taste, and likewise in the case of the Riemann integral in higher dimensional spaces. However, it should be mentioned that when we go deeper into analysis and study the Stieltjes integral it does make a difference which definition of $\succ$ we use. It is not that one is "right" and the other "wrong"; the two definitions lead to different integrals, both of which have been investigated.*

6. Subnets. Now we come back to the question of subnets, postponed a few pages ago. The ordinary definition of subsequence has a straightforward generalization, but this generalization does not prove perfectly satisfactory. J. L. Kelley has proposed an alternative which may at first seem drastic, since it is not merely another scheme for generalizing the idea of subsequence with which we are all familiar, but proposes that we replace that old and familiar idea with a new one. Suppose that f_n, $n=1, 2, \cdots$ is a sequence. We are used to saying that a subsequence of this is a sequence f_{n_j}, $n=1, 2, \cdots$, wherein the subscripts n_j are positive integers such that $n_1<n_2<n_3< \cdots$. But if we look at the proofs of the theorems on subsequences, we find that this last condition is not used in its full strength. What is used is the consequence that as j tends to ∞, so does n_j. Accordingly, Kelley defined a sequence f_{n_j} to be a subsequence of f_n if the n_j are positive integers such that as j tends to ∞ so does n_j. The generalization to nets is obvious. If $f(a)$, a in A is a net, another net $g(b)$, b in B is a subnet of the first if there is a function $a=a(b)$, b in B such that for all b in B, $g(b)=f\{a(b)\}$, and such also that for each a' in A there is a b' in B such that whenever $b \succ b'$ it is also true that $a(b) \succ a'$.

The principal uses of subsequences come by way of the theorem that if a sequence converges, every subsequence converges to the same limit. This gen-

* See, for example, L. M. Graves, *Theory of functions of real variables*, McGraw-Hill, 1946, pp. 260–261.

eralizes at once to nets. As an application, which in spite of its simplicity may serve to hint that our nets may have caught some strange fish in addition to what we wanted and expected, we consider a function $f(x)$, x in A defined on a set A of real numbers, and we assume that $\lim_{x\to c} f(x)$ exists and is equal to k. Let $x_1, x_2, x_3, \cdots$ be a sequence of points of A all different from c and having c as limit when $n\to\infty$. We *could* follow the familiar proof that then $f(x_n)$ has k as limit. But we need not; for $f(x_n)$, $n=1, 2, \cdots$ is by our definition a subnet of the net $f(x)$, x in A (wherein as before we define $a>a'$ to mean $0<|a-c|<|a'-c|$), so the subnet must have k as limit.

7. Relations to topology. So far our discussion has been restricted to topics that an undergraduate might encounter. Now we shall touch briefly on some less elementary uses of Moore-Smith convergence. But first we introduce an abbreviation. If a statement involving members a of a directed set A is true for all a which follow some a' in A, we say that the statement is "eventually" true. Thus we have a clear concept to replace the idea of "becomes and remains" often mentioned in elementary texts. It is in fact convenient to use this idea of "eventually" even in the beginning of the study of Moore-Smith limits. If a statement P is eventually true (say for $a>a'$) and another statement Q is eventually true (say for $a>a''$), we choose a^* such that $a^*>a'$ and $a^*>a''$; then P and Q are both true for $a>a^*$, so the joint statement "P and Q" is eventually true.

A topological space is a set X together with a specified collection of subsets of X, called the "open sets," such that the union of arbitrarily many open sets is open, the intersection of finitely many open sets is open, and the empty set and X itself are open. A neighborhood of a point x of X is an open set containing x. A net $x(a)$, a in A of points of X converges to a point x' of X if for every neighborhood of x', $x(a)$ is eventually in that neighborhood. A point x' is a *cluster point* of the net $x(a)$, a in A if, crudely stated, $x(a)$ "keeps coming back" to every neighborhood of x'; precisely,† for each neighborhood U of x' and each a' in A, $x(a)$ is in U for some $a>a'$.

It is well known that if X is a subset of a euclidean space, either it has all three of the following properties or else it has none of them:

(I) *From every covering of X by open sets it is possible to extract finitely many sets which cover X.*

(II) *Every sequence of points of X has a cluster point in X.*

(III) *Every sequence of points of X contains a subsequence which converges to a point of X.*

These three properties continue to be equivalent for some spaces more general than subsets of euclidean spaces, for example they are equivalent if X is per-

† For this relationship between U and the net $x(a)$, a in A, Halmos proposes the terminology "x is frequently in AU."

fectly separable; but they are not equivalent for all topological spaces. However, as soon as we replace sequences by nets the equivalence is restored. That is, if we write

(II′) *Every net of points of X has a cluster point in X,*

(III′) *For every net $x(a)$, a in A, of points of X there is a subnet converging to a point of X,*

then every topological space X either has all three properties (I), (II′), (III′), or it lacks all three. Suppose that X has property (I), and let $x(a)$, a in A be a net of points of X. If this had no cluster point, for each x in X we could find a neighborhood $U(x)$ such that $x(a)$ is eventually out of $U(x)$. Finitely many of these cover X, say $U(x_1), \cdots, U(x_n)$. For $a \succ$ a certain a_{n_j}, $x(a)$ is not in $U(x_j)$. Choose $a \succ a_1, \cdots, a_n$; then $x(a)$ is out of all the $U(x_j)$ and yet is in X, which is impossible. So X has property (II′). Conversely, suppose X lacks property (I). Then there is a collection K of open sets covering X but such that no finite subcollection of K covers X. Let A consist of all finite subsets of K, and order A by defining $a \succ a'$ to mean that all the sets which belong to a' also belong to a. A is directed by $\succ$, since if a' and a'' are in A, the set a consisting of all sets in a' together with all sets in a'' satisfies $a \succ a'$ and $a \succ a''$. For each such a, there is a point of X not in any of the sets which constitute a, since these sets form a finite subcollection of K and by hypothesis do not cover X. Pick such a point and call it $x(a)$. Because A is directed, these $x(a)$, a in A form a net. Now for any x' in X and any set U of the family K which contains x', we let a' be the subset of K consisting of U alone. If $a \succ a'$, then $x(a)$ is outside all the sets in a, in particular is outside U, so x' is not a cluster point of the net. This holds for all x' in X, so X lacks property (II′).

Property (III′) plainly implies (II′). To show the converse,* let $x(a)$, a in A be a net of points of X having x' as cluster point. Let B consist of all the pairs (U, a) in which U is a neighborhood of x' and a is a point of A such that $x(a)$ is in U. We order these by defining $(U, a) \gg (U', a')$ to mean that U is contained in U' and $a \succ a'$. This is obviously transitive. If (U', a') and (U'', a'') are in B, let U be the intersection of U' and U''. This is a neighborhood of x', so there is an a in A such that $a \succ a'$ and $a \succ a''$ and $x(a)$ is in U. So by definition $(U, a) \gg (U', a')$ and $(U, a) \gg (U'', a'')$, and B is directed by $\gg$. For each $b = (U, a)$ in B we define $a(b)$ to be the second component of b, that is $a(b) = a$. We must prove, first, that $x\{a(b)\}$, b in B is a subnet of $x(a)$, a in A, and second, that it converges to x'. We do both of these at once, as follows. Let a' be an arbitrary member of A and U' an arbitrary neighborhood of x'. There is an a'' in A such that $a'' \succ a'$ and $x(a'')$ is in U'. Define $b' = (U', a'')$. For every $b = (U, a)$ such that $b \gg b'$, we have first $a \succ a'' \succ a'$, that is $a(b) \succ a'$, whence $x\{a(b)\}$, b in B is a subnet of $x(a)$, a in A. And second we have U contained in

* J. L. Kelley, *loc. cit.*

U', so that $x\{a(b)\}$, being in U, is also in U', whenever $b \gg b'$; and therefore x' is the limit of $x\{a(b)\}$, b in B. This completes the proof of the theorem. It will be noticed that Kelley's definition of subnet is exactly what is needed; under previous definitions of subnet the theorem cannot be established.

When we are dealing with subspaces of finite dimensional spaces, it is quite convenient to be able to maneuver back and forth between property (I), in "Heine-Borel" arguments, and properties (II) and (III), in "Bolzano-Weierstrass" arguments. The proof just completed shows that the same kind of convenience is available even in the most general topological spaces, provided that we agree to use nets instead of sequences and to adopt Kelley's definition of subsequence.

8. Convergence in partially ordered spaces. In the earlier pages of this paper we were concerned with sets of real numbers and nets of real numbers. The real numbers have a natural topology, and if we think of the reals as a space of points equipped with this topology the natural generalization is to topological spaces. On the other hand, the reals also constitute a partially ordered, in fact a directed, system. It is thus also reasonable to wonder how much we can deduce from the order relation, and to investigate partially ordered sets and nets of points of partially ordered sets. Suppose then that X is a set of points partially ordered by a relation $\succ$. To save trouble we shall assume that $\succ$ is a *proper* partial ordering, which by definition means that if x and x' are two different points of X, the relation $x \succ x'$ and $x' \succ x$ cannot both be true. For example, when in discussing integrals we partially ordered partitions by defining $P \succ P'$ to mean that norm $P \leqq$ norm P', we thereby introduced an improper partial ordering. But if we had instead defined $P \succ P'$ to mean norm $P <$ norm P' the partial ordering would have been proper.

The definitions of upper and lower bounds and of least upper bounds (or suprema) and greatest lower bounds (or infima) can be taken over at once from the real numbers. Among the reals, there are two equivalent ways of defining completeness. The reals are complete in the Cauchy sense—every sequence of reals which satisfies the Cauchy condition is convergent to a real limit. They are also complete in the Dedekind sense—every non-empty set of real numbers which has an upper bound has a least upper bound, and likewise for lower bounds. The former of these is of course much used in metric spaces; it is the other which we now consider. We could of course take it over verbatim, and say that a properly partially ordered set is Dedekind-complete if every non-empty subset which has an upper bound also has a supremum. But a directed set with this property would then have a supremum and an infimum for each two points, and by definition such a set is a lattice. Now lattices are interesting objects to study, as anyone knows who has looked into Birkhoff's book about them.* But at this moment we do not wish to confine our attention to them. So we

* Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. XXV, 1948.

adopt another formulation which makes no difference when we are discussing the reals, but makes just the difference we want in other cases. We say that X is Dedekind-complete* if every non-empty subset S of X which *is directed by* $\succ$ and has an upper bound also has a supremum, and every non-empty subset S of X which is directed by $\succ$ and has a lower bound also has an infimum. Thus for example, the set of circular regions of the plane $(x-x_0)^2+(y-y_0)^2 \leqq r^2$, $r \geqq 0$ is Dedekind-complete if $\succ$ means $\supset$, "contains"; but these circles do not form a lattice.

If $f(a)$, a in A is a net of real numbers, the statement that the limit of $f(a)$ is k can be thus phrased in terms of order: For every real number $m<k$, it is eventually true that $f(a)>m$; and for every real number $n>k$, it is eventually true that $f(a)<n$. This suggests the following definition of convergence in partially ordered sets. If $f(a)$, a in A is a net of points of a partially ordered set X, it is convergent if there exist sets M and N in X, directed by $\succ$ and $\prec$ respectively, such that the supremum of M is the same as the infimum of N, and for every m in M and every n in N it is eventually true that $n \succ f(a) \succ m$. It is easy to show that the limit, if it exists, is unique. From the definition of limit we can proceed to the definition of continuity and to the study of continuous functions. Just where this leads us cannot yet be stated. I have done some studying of partially ordered sets and continuous functions on them, and expect to publish the results soon. But much remains to be done.

ON THE RECTANGULAR HYPERBOLA

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1. Introduction. By proper choice of axes and unit of scale any rectangular hyperbola may be expressed by the equation $xy=1$, or, parametrically, by $x=t$, $y=t^{-1}$. Choose any three points A_1, A_2, A_3 on the hyperbola such that no side of the triangle is parallel to an asymptote, and designate the parameters of these points by t_1, t_2, t_3, respectively.

If the altitude of the triangle drawn from A_1 to the side A_2A_3, produced if necessary, makes acute angles α and β with the asymptotes of the hyperbola, then the line drawn through A_1 which makes acute angles β and α, respectively, with the asymptotes is called the antiparallel to the altitude with respect to the asymptotes. For brevity in this article, such a line will be termed a slant-line. Thus the antiparallel to the altitude through A_i with respect to the asymptotes of the hyperbola will be called a slant-line and designated by the symbol H_i, $(i=1, 2, 3)$.

* [Added in proof.] Since writing this, I have learned that this concept, with isotone sequences in place of directed sets, has been thought of earlier; see M. H. Stone, *A general theory of spectra*, I, Proc. Nat. Acad. Sci. 26 (1940), p. 281, and S. Bochner, *Completely monotone functions*, Duke Math. Jour. 9 (1942), p. 520.

A second type of line is one that forms with a given line an isosceles triangle with respect to an asymptote of the hyperbola. Such a pair of lines shall be called *isosceles lines*, and it is evident that a pair of lines isosceles with respect to one asymptote is isosceles with respect to the other.

2. The point H′. The equation of the side A_2A_3 is $x+t_2t_3y=t_2+t_3$; the equation of the altitude through A_1 is $t_2t_3x-y=s_3-t_1^{-1}$; the equation of the slant-line H_1 is $x-t_2t_3y=t_1-t_2t_3t_1^{-1}$. Note that A_2A_3 and H_1 are isoseceles lines. Now the point H' whose coordinates are $(s_1, s_2s_3^{-1})$, where the s_i are the elementary symmetric functions of t_1, t_2, and t_3, lies on the line H_1, since its coordinates satisfy the equation of H_1. Moreover, since the coordinates of H' are symmetric in the parameters t_i, $(i=1, 2, 3)$, it follows that the three slant-lines H_1, H_2, and H_3 are concurrent at H'.

The line H_1 cuts A_2A_3 at the point C_1 whose x-coördinate is $\frac{1}{2}(s_1-t_2t_3t_1^{-1})$; it cuts the hyperbola at B_1 with x-coördinate $-t_2t_3t_1^{-1}$. It is obvious that C_1 is the midpoint of $H'B_1$, whence we have the theorem that *the images of H' in the sides of the triangle $A_1A_2A_3$ measured along the slant-lines of $A_1A_2A_3$ lie on the hyperbola.*

The triangle $A_1A_2A_3$ has a centroid G and a center O of its circumscribed circle. Locate a point H on the line OG such that $OH=3OG$. The lines joining H to the vertices A_i are the altitudes of the triangle $A_1A_2A_3$ and thus H is its orthocenter and lies on the hyperbola [1].

Let us determine a point K on the line joining C, the center of the hyperbola, to G, such that $CK=3CG$. The coördinates of this point K turn out to be $(s_1, s_2s_3^{-1})$ or those of H'. The lines joining H' to the vertices A_i are the slant-lines H_i, and we shall call H' the slant-center of the triangle $A_1A_2A_3$. Moreover H' lies on the circumcircle of $A_1A_2A_3$.

3. The line of slant-images. The slant-line of any point T on the hyperbola, with parameter T, drawn to the side A_2A_3, has the equation $x-t_2t_3y=T-t_2t_3T^{-1}$. The image of T in A_2A_3 measured along this line—let us call it the slant-image of T in A_2A_3—has for its x-coördinate $x=t_2+t_3-t_2t_3T^{-1}$. The slant-image of T in the side A_3A_1 has for its x-coördinate the value $x=t_3+t_1-t_3t_1T^{-1}$. The equation of the line joining these two slant-images is

$$Tx - s_3y = Ts_1 - s_2. \tag{1}$$

From symmetry it is evident that the slant-image of T in the side A_1A_2 likewise lies on this line. We note that the coördinates of the point H' satisfy equation (1). Hence, *the three slant-images of any point on the hyperbola as to $A_1A_2A_3$ lie on a straight line. As T runs over the hyperbola, we obtain a pencil of lines on H'.*

Given any line on H', to find the point T on the hyperbola for which it is the line of slant-images, we may proceed as follows: draw through any vertex A_i a line parallel to the given line which intersects the hyperbola at a second point V_i, the further intersection of the slant-line through V_i, parallel to H_i,

will cut the hyperbola at the point T.

The diametrically opposite point on the hyperbola to the point T has the coördinates $(-T, -T^{-1})$. The equation of its line of slant-images is

$$Tx + s_3y = Ts_1 + s_2. \tag{2}$$

Hence *the slant-image lines of any two diametrically opposite points on the hyperbola are isosceles lines.*

The lines (1) and (2) cut the side A_2A_3 in two points whose midpoint D_1 is found to have the x-coördinate $(s_3+s_1T^2)(T^2-t_1^2)^{-1}$. Similarly the lines (1) and (2) cut the side A_3A_1 in two points whose midpoint D_2 has the x-coördinate $(s_3+s_1T^2)(T^2-t_2^2)^{-1}$. Now the line joining D_1 to D_2 has the equation

$$T^2(s_2 + T^2)x + s_3(s_3 + s_1T^2)y = (s_2 + T^2)(s_3 + s_1T^2). \tag{3}$$

From symmetry, the point D_3, the midpoint of the segment cut on A_1A_2 by lines (1) and (2), must lie on this line. Hence, *the midpoints of the segments cut on the sides of the triangle $A_1A_2A_3$ by the lines of slant-images of any two diametrically opposite points of the hyperbola lie on a line.*

The line joining H' to D_1 has the equation

$$T^2x + t_1s_3y = s_1T^2 + t_3s_2.$$

Now the line isosceles to $H'D_1$ through A_1 is

$$T^2x - t_1s_3y = T^2t_1 - s_3. \tag{4}$$

This equation obviously is satisfied by $x=-s_3T^{-2}$, $y=-T^2s_3^{-1}$, the coördinates of a point R on the hyperbola. Since the coördinates of R are symmetric in t_1, t_2 and t_3 we can state that *the lines through A_i isosceles to $H'D_i$ meet at a point R on the hyperbola.*

4. The nine-point hyperbola. It has been shown that the points where the slant-lines H_i cut the sides A_jA_k, namely, the points C_i, the midpoints E_i of the sides of $A_1A_2A_3$, and the midpoints of the segments A_iH', all lie on a rectangular hyperbola whose asymptotes are parallel to the asymptotes of the given hyperbola and whose transverse axis is one half as long [2]. In our notation the equation of this nine-point hyperbola is

$$(2x - s_1)(2s_3y - s_2) = s_3. \tag{5}$$

The center of this hyperbola is $\frac{1}{2}(s_1, s_2s_3^{-1})$, a point N' lying on the line joining the center C of the given hyperbola to the point H' such that $CN'=N'H'$. The centroid G of the triangle $A_1A_2A_3$ lies also on this line since $2CG=GH'$. Thus there exists another line, analogous to the Euler line connected with a triangle, on which lie four harmonic points, namely, C the center of the given hyperbola, G the centroid of $A_1A_2A_3$, N' the center of the nine-point hyperbola, and H' the meet of the slant-lines of $A_1A_2A_3$.

The line joining C_2C_3 has the equation

$$2t_2t_3(x + t_1^2y) = t_2t_3s_1 + t_1s_2 - t_1t_2^2 - t_1t_3^2.$$

This line is parallel to the tangent to the hyperbola at A_1 and also parallel to the tangent to the nine-point hyperbola at E_1, the midpoint of the side A_2A_3.

5. Lines related to a diameter. The equation of the line joining the points with x-coördinates T and $-T$ is given by

$$x - T^2y = 0. \tag{6}$$

The line through A_1 isosceles to (6) is

$$x + T^2y = t_1 + T^2t_1^{-1}.$$

This cuts the diameter (6) at F_1 whose x-coördinate is $\frac{1}{2}(t_1+T^2t_1^{-1})$. The equation of the slant-line through F_1, parallel to H_1, is

$$x - t_2t_3y = \tfrac{1}{2}(t_1 - t_2t_3t_1^{-1} + T^2t_1^{-1} - s_3T^{-2}). \tag{7}$$

The equation of the slant-line through F_2, parallel to H_2, is

$$x - t_3t_1y = \tfrac{1}{2}(t_2 - t_3t_1t_2^{-1} + T^2t_2^{-1} - s_3T^{-2}). \tag{8}$$

The lines (7) and (8) meet at the point whose x-coördinate is

$$x = \tfrac{1}{2}(s_1 - s_3T^{-2}). \tag{9}$$

Through this point must pass likewise the slant-line through F_3 parallel to H_3. Hence, *given any diameter of the hyperbola, the lines through A_i isosceles to the diameter meet the diameter in three points F_i. The slant-lines through F_i, parallel to H_i, meet at a point on the nine-point hyperbola, the midpoint of $H'R$.*

The line through A_1, parallel to the diameter (6) of the hyperbola, cuts the hyperbola again at L_1 whose x-coördinate is $-T^2t_1^{-1}$. The diameter (6) cuts the side A_2A_3 at Q_1 whose x-coördinate is $T^2(t_2+t_3)(T^2+t_2t_3)^{-1}$. The equation of L_1Q_1 is

$$t_1(s_1T^2 + s_3)x - T^4(s_2 + T^2)y = T^2(t_2 + t_3)(t_1^2 - T^2). \tag{10}$$

The second intersection of L_1Q_1 with the hyperbola is the point whose x-coördinate is

$$x = \frac{T^2(s_2 + T^2)}{s_3 + s_1T^2}. \tag{11}$$

Hence, we can state that *if the parallels through A_i to any diameter of the hyperbola cut the hyperbola again at points L_i, and if the diameter cuts the sides of the triangle $A_1A_2A_3$ at points Q_i, the three lines L_iQ_i meet at a point on the hyperbola whose line of slant-images is isosceles to the line* (3).

The equation of the slant-line through L_1, parallel to H_1, is

$$x - t_2t_3y = -T^2t_1^{-1} + s_3T^{-2}.$$

The equation of the slant-line through L_2, parallel to H_2, is

$$x - t_3t_1y = -T^2t_2^{-1} + s_3T^{-2}.$$

These two lines intersect at $R'(s_3T^{-2}, T^2s_3^{-1})$, a point on the hyperbola diagonally opposite to point R, mentioned in Section 3. Naturally the slant-line through L_3, parallel to H_3, is on R'. Hence, *the three slant-lines through L_i parallel to H_i meet at a point R' on the hyperbola.*

The following theorem is stated without proof: *The parallel through H' to A_1R' meets the side A_2A_3 at D_1. Hence the three points in which parallels through H' to A_iR' meet the sides A_jA_k are collinear and lie on the line* (3).

6. The lines isosceles to any line. The equation of any line in the plane not parallel to any asymptote may be written as

$$ax + by = c, \qquad a = b \neq 0. \tag{12}$$

The isosceles line to (12) through A_1 cuts (12) at the point with coördinates $1/2a[at_1 - bt_1^{-1} + c]$, $1/2b[bt_1^{-1} - at_1 + c]$. The equation of the slant-line through this point parallel to H_1 is

$$x - t_2t_3y = \frac{1}{2a}[at_1 - bt_1^{-1} + c] - \frac{1}{2b}[bt_2t_3t_1^{-1} - as_3 + t_2t_3c].$$

On this line lies the point $1/2(s_1 + ca^{-1} + ab^{-1}s_3)$, $1/2(s_2s_3^{-1} + b^{-1}c + a^{-1}bs_3^{-1})$. Hence, *given the points N_i where the lines isosceles to $ax+by=c$ through the vertices A_i cut the given line $ax+by=c$, the three slant-lines through N_i parallel to H_i meet in a point.* For the diameter $x - T^2y = 0$ this point is (9), a point on the nine-point hyperbola.

7. Sets of four points. The four points A_1, A_2, A_3 and their orthocenter $H(-s_3^{-1}, -s_3)$ which lie on the hyperbola are a set of orthic points, *i.e.*, any point is the orthocenter of the triangle formed by the other three points. All four triangles have the same nine-point circle [3].

The four points A_1, A_2, A_3, and H' which lie on the circumcircle of A_1, A_2, A_3 are a set of slant points, *i.e.*, any point is the slant-center of the triangle formed by the other three points. All four triangles have the same nine-point hyperbola.

The four points A_1, A_2, A_3 and S which lie on both the circumcircle and the hyperbola have as orthocenters for the four triangles formed from these points, the points S_i, and H. It is known [4] that these points lie on the given hyperbola, diametrically opposite to A_i and S, and also on a circle with center at

$$x = -\tfrac{1}{2}(s_1 + s_3^{-1}),$$

and with radius equal to the radius of the circumcircle. It follows immediately

that the orthocenters of the four triangles formed from the points S_i and H are the original points A_i and S.

The four points A_1, A_2, A_3 and S have as slant-centers, for the four triangles formed from these points, the points S_i' and H'. These four points lie on the circumcircle diametrically opposite to A_i and S, and also on a hyperbola, whose equation is

$$(x - s_1 - s_3^{-1})(y - s_2 s_3^{-1} - s_3) = 1.$$

The center of this hyperbola C' lies on the line CO, such that $CO = OC'$. It follows at once that the slant-centers of the four triangles formed from the points S_i' and H' are the points A_i and S. Consequently, *the four points $A_1A_2A_3$, S, the intersections of the circumcircle and the given hyperbola, have as orthocenters a set of four points whose orthocenters again are the original points; likewise they have as slant-centers a set of four points whose slant-centers are the original points.*

8. The pedal line of the hyperbola. The line through any point T of the hyperbola parallel to the slant-line H_1 cuts the side A_2A_3 in a point whose x-coördinate is given by $2x = t_1 + t_2 + T - t_1t_2T^{-1}$. Three such points, each on a side of the triangle $A_1A_2A_3$, lie on the line whose equation is

$$Tx - s_3y = \tfrac{1}{2}(T^2 + s_1T - s_2 - s_3T^{-1}). \tag{13}$$

We shall call this line the pedal line of the point T. The pedal line of the diametrically opposite point to T on the hyperbola is

$$Tx + s_3y = \tfrac{1}{2}(-T^2 + s_1T + s_2 - s_3T^{-1}). \tag{14}$$

These lines intersect at $x = \frac{1}{2}(s_1 - s_3T^{-2})$, the coördinate of a point on the nine-point hyperbola which is the midpoint of the slant-center H' and the point R of Section 3. The equation of the lines joining R to T and to T', the diametrically opposite point of T on the hyperbola are, respectively,

$$Tx - s_3y = T^2 - s_3T^{-1},$$

and

$$Tx + s_3y = -T^2 - s_3T^{-1}.$$

Hence, the *angle between the pedal lines of any two diametrically opposite points T and T' of the hyperbola equals the angle TRT' at the point R on the hyperbola.*

Given any four points A_1, A_2, A_3, A_4 of the hyperbola, the four nine-point hyperbolas of the four triangles formed from the four points have a common point. Thus we see that the equation of the nine-point hyperbola of $A_1A_2A_3$, namely,

$$(2x - s_1)(2s_3y - s_2) = s_3,$$

is satisfied by the coördinates of the point $(\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_3\sigma_4^{-1})$, where the σ_i are symmetric functions of the four t_1, t_2, t_3, t_4. From symmetry this point must lie on the other three nine-point hyperbolas.

The pedal line of the point A_4 as to the triangle $A_1A_2A_3$ is

$$t_4x - s_3y = \tfrac{1}{2}(t_4^2 + s_1t_4 - s_2 - s_3t_4^{-1}), \tag{15}$$

which is also on the point $(\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_3\sigma_4^{-1})$. This point is the midpoint of the join of A_i to the slant-center of the other three points. Hence, *the four pedal lines of A_i as to the triangle $A_jA_kA_l$ are concurrent.* Given *any four points A_1, A_2, A_3, A_4 on the hyperbola, from which we can form four triangles. Any point T of the hyperbola will determine four pedal lines, one for each triangle, the lines isosceles to each of these lines from T meets each line in four collinear points,* for the pedal line of T as to $A_1A_2A_3$ is

$$Tx - s_3y = \tfrac{1}{2}(T^2 + s_1T - s_2 - s_3T^{-1}).$$

The line through T, isosceles to this line is

$$Tx + s_3y = T^2 + s_3T^{-1}.$$

Their point of intersection has for x-coördinate

$$4x = 3T + s_1 - s_2T^{-1} + s_3T^{-2}.$$

Four such points lie on the line

$$T^2x + \sigma_4y = \tfrac{1}{4}(3T^3 + \sigma_1T^2 - \sigma_2T + \sigma_3 + \sigma_4T^{-1}),$$

where the σ_i refer to symmetric functions of t_1, t_2, t_3, t_4. The pedal lines of any two points T_1 and T_2 with respect to the triangle $A_1A_2A_3$ meet at the point M whose x-coördinate is given by

$$2x = T_1 + T_2 + s_1 + s_3T_1^{-1}T_2^{-1}. \tag{16}$$

Now the pedal line of the point $T_3(s_3T_1^{-1}T_2^{-1}, T_1T_2s_2^{-1})$ is also on the point (16), and since T_3 is on the hyperbola we see that for the triangle $A_1A_2A_3$ there are three points T_1, T_2, T_3 whose pedal lines are concurrent. *The point of concurrence is midway between the slant-centers of the two triangles.* To locate T_3, given T_1, T_2 and the triangle $A_1A_2A_3$, we construct the slant-center H' of $A_1A_2A_3$. The pedal lines of T_1 and T_2 as to $A_1A_2A_3$ intersect at M. Produce $H'M = MY$, then Y is the slant-center of T_1, T_2 and T_3. Also the pedal line of T_i and the join of T_jT_k are isosceles lines.

Given, *any four points $A_1A_2A_3A_4$ on the hyperbola, which will form four triangles. Points T_1 and T_2 on the hyperbola will each have four pedal lines with respect to each triangle; each pair will intersect in a point, thus giving rise to four points which lie on a line.* We saw that the pedal lines of any two points T_1 and T as to the triangle $A_1A_2A_3$ met at

$$2x = T_1 + T_2 + s_1 + s_3T_1^{-1}T_2^{-1}.$$

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The four such points arising from the four triangles satisfy the equation

$$(17)\quad 2(T_1T_2x + \sigma_4 y) = T_1T_2\sigma_1 + \sigma_3 + T_1^2T_2 + T_1T_2^2 + \sigma_4(T_1 + T_2)T_1^{-1}T_2^{-1},$$

where the σ_i are symmetric functions of t_1, t_2, t_3, t_4. For any third point T_3 on the hyperbola, with reference to $A_1A_2A_3A_4$, we shall have three such lines as (17), taking the points T_i by pairs. The three lines meet at a point M' whose x-coördinate is given by

$$2x = \sigma_1 + T_1 + T_2 + T_3 - \sigma_4 T_1^{-1}T_2^{-1}T_3^{-1}.$$

Since this is symmetric in the t_i and T_i, we have the theorem that *the three lines* (17) *for* T_1T_2, T_2T_3 *and* T_3T_1 *as to* $A_1A_2A_3A_4$ *are concurrent.*

9. The eighteen-point hyperbola. If hyperbolas* be drawn with centers at C_i, the feet of the slant-lines, and passing through C, the center of the hyperbola, they cut the corresponding sides of triangle $A_1A_2A_3$ in six points lying on a hyperbola whose center is at H'.

The equation of the hyperbola with center at C_3 and on C is

$$(2t_3x - t_3s_1 + t_1t_2)(2s_3y - s_2 + t_3^2) = (s_2 - t_3^2)(t_3s_1 - t_1t_2).$$

This cuts the side A_1A_2 in two points whose coördinates are given by

$$(18)\qquad 2t_3x = t_3s_1 - t_1t_2 \pm a^{1/2}, \qquad 2s_3y = s_2 - t_3^2 \mp a^{1/2},$$

where

$$a = (s_2 - t_3^2)(t_1t_2 - t_3s_1).$$

The two points (18) lie on the hyperbola

$$(19)\qquad 2(x - s_1)(s_3y - s_2) = s_1s_2 + s_3.$$

Since this equation is symmetric in $t_i (i = 1, 2, 3)$, it follows that the points cut on the sides A_2A_3 and A_3A_1 by hyperbolas with centers at C_1 and C_2, respectively, and on C, likewise lie on (19).

Secondly, if hyperbolas be drawn with centers at the vertices A_i of the triangle $A_1A_2A_3$, and with transverse axis equal to the given hyperbola, they cut the lines joining the midpoints of the sides A_iA_j and A_iA_k, respectively, in six points lying on (19).

The equation of the hyperbola, with transverse axis equal to the given hyperbola and with center at A_2, is

$$(x - t_2)(t_2y - 1) = t_2.$$

* The word hyperbola, in this section, refers to a rectangular hyperbola with asymptotes parallel to those of the given hyperbola.

The equation of the line joining the midpoints of A_2A_3 and A_2A_1 is

$$2(t_2x + s_3y) = s_2 + t_2^2.$$

The coördinates of the two points of intersection of this line and hyperbola are given by

$$4t_2x = s_2 + 3t_2^2 - 2t_1t_3 \mp b^{1/2}, \qquad 4s_3y = s_2 - t_2^2 + 2t_1t_3 \pm b^{1/2} \tag{20}$$

where

$$b = (s_2 - t_2^2 + 2t_1t_3)^2 - 8t_1t_3s_2 - 8t_2s_3.$$

By substituting the values given in (20) in equation (19) one can show (19) is satisfied.

Thirdly, if hyperbolas be drawn with centers at the midpoints of A_iH' and passing through C, they cut the lines parallel to A_jA_k, and on these midpoints, in six points lying on (19). This follows from the fact that the hyperbola,

$$(2x - s_1 - t_3)(2s_3y - s_2 - t_1t_2) = (s_1 + t_3)(s_2 + t_1t_2),$$

meets the line,

$$t_3x + s_3y = s_2 + t_3^2,$$

in the two points given by

$$2t_3x = t_3s_1 + t_3^2 \pm c^{1/2}, \qquad 2s_3y = 2s_2 + t_3^2 - t_3s_1 \mp c^{1/2}, \tag{21}$$

where

$$c = t_3^2s_1^2 - t_3^4 - 2t_3s_1s_2 - 2t_3^2s_2.$$

The coördinates of the points (21) will satisfy equation (19). Hence this hyperbola passes through eighteen points, two on each side of triangle $A_1A_2A_3$, two on each side of the lines joining the midpoints of the sides of $A_1A_2A_3$, and two on each line parallel to the sides of $A_1A_2A_3$ and on the midpoints of A_iH'.

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L'HOSPITAL'S RULE

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1. Prefatory remarks. In this article I have assembled some material pertaining to l'Hospital's rule which it seem to me may be of interest to teachers of analysis. The immediate stimulus for the writing of the paper was my discovery of a new proof which has the virtue of disposing completely of the rule in brief compass, without any need for tedious changes of variable or the treatment of a variety of special instances. The material of §3 is not new, but is probably unfamiliar to many readers of the MONTHLY.

2. The unified theorem and its proof. Let I be an open interval of the x-axis. We understand that I may be either bounded or unbounded. Let c be one of the extremities of I. If I is bounded, c is a real number, but if I is unbounded, c may be $+\infty$ or $-\infty$. L'Hospital's rule is concerned with two functions $f(x)$, $g(x)$, which are real-valued and defined on I. The first additional hypotheses of f and g are the following:

H_0: *The derivatives* $f'(x)$, $g'(x)$ *are defined (as finite limits) on* I. *Furthermore,* $g(x)$ *and* $g'(x)$ *are never zero on* I.

There are two cases of l'Hospital's rule, depending on which of the two following hypotheses are fulfilled.

H_1: $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0.$$

H_2: $$\lim_{x\to c} |g(x)| = \infty.$$

We are now ready to state the unified theorem.

THEOREM I. *Let* A *denote either a real number or one of the symbols* $+\infty$, $-\infty$. *Suppose that* f, g *satisfy hypothesis* H_0, *either one of the two hypotheses* H_1, H_2, *and the further hypothesis*

H_3: $$\lim_{x\to c} \frac{f'(x)}{g'(x)} = A.$$

Then we can conclude that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = A. \tag{2.1}$$

Theorem I (l'Hospital's rule) is obviously a corollary of the following more general proposition:

THEOREM II. *Let* f, g *satisfy* H_0 *and either* H_1 *or* H_2. *Then*

$$\underline{\lim_{x\to c}} \frac{f'(x)}{g'(x)} \leqq \underline{\lim_{x\to c}} \frac{f(x)}{g(x)} \leqq \overline{\lim_{x\to c}} \frac{f(x)}{g(x)} \leqq \overline{\lim_{x\to c}} \frac{f'(x)}{g'(x)}. \tag{2.2}$$

For the proof of Theorem II we start with Cauchy's formula (the extended law of the mean)

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(X)}{g'(X)}, \tag{2.3}$$

where x and y are any distinct points of I, and X is some point between x and y; that $g(x) \neq g(y)$ follows from H_0. Let x be in the interval I, and define

$$m(x) = \text{g.l.b.} \frac{f'(\xi)}{g'(\xi)}, \qquad M(x) = \text{l.u.b.} \frac{f'(\xi)}{g'(\xi)} \tag{2.4}$$

for ξ between x and c. There are of course the possibilities that $m(x) = -\infty$ and that $M(x) = +\infty$. We take y between x and c; then, by (2.3) and (2.4) we may write

$$m(x) \leqq \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(y)}{g(x)}}{1 - \dfrac{g(y)}{g(x)}} \leqq M(x) \tag{2.5}$$

and

$$m(x) \leqq \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} \leqq M(x). \tag{2.6}$$

Now let $y \to c$. If H_1 holds, we use (2.5) and obtain

$$m(x) \leqq \frac{f(x)}{g(x)} \leqq M(x).$$

If H_2 holds, we use (2.6) and obtain

$$m(x) \leqq \varliminf_{x \to c} \frac{f(y)}{g(y)} \leqq \varlimsup_{x \to c} \frac{f(y)}{g(y)} \leqq M(x).$$

We then let $x \to c$ and the desired conclusion (2.2) follows in either case.

Comment on the proof. The Polish mathematician T. Wazewski has given interesting proofs of Theorems I and II, using as new ingredient a lemma about pairs of sequences of real numbers [4 and 5]. Wazewski's arguments suggested to me the principle of the proof given here. My arrangement of the proof has been somewhat shortened and improved by the suggestions of the referee.

3. Other proofs. Fritz Lettenmeyer [1] has devised a wholly different method of proof which can be applied to all cases of l'Hospital's rule. The method does not make use of Cauchy's formula at all. This proof does not seem to be widely known. I shall give the proof of Theorem I with hypothesis H_2 on the assumption that A is finite and $c=+\infty$.

The derivative $g'(x)$ must be of constant sign, since its never vanishes; this is a consequence of Darboux's theorem (see [3], p. 637). We may then assume $g'(x)>0$; otherwise we consider the ratio of $-f(x)$ to $-g(x)$. It follows from H_2 that $\lim_{x\to+\infty} g(x)=+\infty$. With $\epsilon>0$ given we have

$$A-\epsilon<\frac{f'(x)}{g'(x)}<A+\epsilon$$

for all sufficiently large values of x. Then

$$(A-\epsilon)g'(x)<f'(x)<(A+\epsilon)g'(x),$$

whence it follows that $f(x)-(A-\epsilon)g(x)$ is monotone increasing and $f(x)-(A+\epsilon)g(x)$ is monotone decreasing. These same conclusions hold with ϵ replaced by $\epsilon/2$. Now

$$f(x)-\left(A-\frac{\epsilon}{2}\right)g(x)<f(x)-(A-\epsilon)g(x). \tag{3.1}$$

We infer that

$$\lim_{x\to+\infty}\left[f(x)-(A-\epsilon)g(x)\right]=+\infty, \tag{3.2}$$

for if this is not so, the monotone function $f(x)-(A-\epsilon)g(x)$ has a finite limit, and hence, by (3.1), so has the function $f(x)-(A-\epsilon/2)g(x)$. But then the difference of these functions, namely $\frac{1}{2}\epsilon\, g(x)$, also has a finite limit. This contradicts H_2, however, so that (3.2) holds. In the same way we prove that

$$\lim_{x\to+\infty}\left[f(x)-(A+\epsilon)g(x)\right]=-\infty. \tag{3.3}$$

Therefore, for all sufficiently large values of x,

$$f(x)-(A+\epsilon)g(x)<0<f(x)-(A-\epsilon)g(x),$$

or

$$A-\epsilon<\frac{f(x)}{g(x)}<A+\epsilon.$$

Thus (2.1) is proved.

We leave the proof when $A=\pm\infty$ to the reader. This method of proof can readily be adapted to use with the hypothesis H_1. It would be worth while to cast this method into a form where the consideration of various cases is obviated.

Finally, another group of proofs of l'Hospital's rule is obtained if we take advantage of the fact that $g(x)$ is a monotone function and hence possesses a continuous inverse function. If we let G denote the inverse function, so that $x=G(y)$ is equivalent to $y=g(x)$, and define

$$F(y) = f(G(y)), \tag{3.4}$$

then

$$\frac{f(x)}{g(x)} = \frac{F(y)}{y}, \qquad F'(y) = \frac{f'(x)}{g'(x)}. \tag{3.5}$$

If $b=\lim_{x\to c} g(x)$, G is defined in an open interval J with extremity $y=b$. For the proof under hypothesis H_1 with c finite we may apply the ordinary law of the mean to $F(y)$ (see e.g. [2], p. 177).

We shall use this method to give an alternative proof of Theorem II under hypothesis H_2.

LEMMA. *Let J be an unbounded open interval having one extremity b (where $b=+\infty$ or $-\infty$). Let $F(y)$ be defined and admit a finite derivative $F'(y)$ at each point of J. Then*

$$\overline{\lim_{y\to b}} \frac{F(y)}{y} \leqq \overline{\lim_{y\to b}} F'(y) \tag{3.6}$$

and

$$\underline{\lim_{y\to b}} F'(y) \leqq \underline{\lim_{y\to b}} \frac{F(y)}{y}. \tag{3.7}$$

Under the change of variable $y=g(x)$, $x\to c$ is equivalent to $y\to b$. By virtue of equations (3.5) it is clear that the Lemma is equivalent to Theorem II with hypothesis H_2.

For definiteness assume $b=+\infty$. We may assume

$$B = \overline{\lim_{y\to b}} F'(y) < +\infty,$$

since otherwise there is nothing to prove in (3.6). If $\epsilon>0$ we have

$$F'(y) < B + \frac{\epsilon}{2} < B + \epsilon$$

for all sufficiently large values of y. By reasoning exactly like that in Lettenmeyer's proof given earlier in this section we conclude that

$$\frac{F(y)}{y} < B + \epsilon$$

when y is sufficiently large. This proves (3.6) for $b=+\infty$. A similar proof may be given for (3.7) when $b=+\infty$. To prove (3.7) when $b=-\infty$ we may take the change of variable

$$t=-y, \qquad \Phi(t)=F(-t).$$

Then

$$\overline{\lim_{t\to+\infty}}\ \Phi'(t) = \overline{\lim_{y\to-\infty}}\ \{-F'(y)\} = -\underline{\lim_{y\to-\infty}}\ F'(y),$$

$$\overline{\lim_{t\to+\infty}}\ \frac{\Phi(t)}{t} = -\underline{\lim_{y\to-\infty}}\ \frac{F(y)}{y},$$

so that the inequality

$$\underline{\lim_{y\to-\infty}}\ F'(y) \leqq \underline{\lim_{y\to-\infty}}\ \frac{F(y)}{y}$$

follows from what has already been proved. Likewise we may deduce (3.6) when $b=-\infty$ from (3.7) when $b=+\infty$. This completes the proof of the lemma.

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PERSPECTIVE AND ORTHOLOGIC TRIANGLES AND TETRAHEDRONS*

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1. Introduction. In 1894 P. Sondat published a fundamental theorem concerning two perspective and orthologic triangles [1]. The corresponding theorem for two perspective and orthologic tetrahedrons was stated several years later [2]. These two theorems, which have not been presented to the readers of the MONTHLY previously, have many applications. It is proposed to give an elementary proof of the Sondat theorem in such a manner that it may be generalized, and to cite some interesting particular cases. The word bilogic intro-

* Translated by W. E. Byrne, Virginia Military Institute, (with additional information supplied by M. Thébault in letters dated 13 February and 13 April 1950).

duced by Neuberg [3] will be used to designate triangles (tetrahedrons) which are perspective and orthologic.

2. Sondat's theorem. *If two triangles $T \equiv ABC$, $T_1 \equiv A_1B_1C_1$ are bilogic, the center of perspective P and the orthologic centers Q and Q_1 are on the same line perpendicular to the axis of perspectivity d.*

Proof. By definition of orthologic triangles the point Q is the point common to the perpendiculars drawn through A_1, B_1, C_1 to the sides BC, CA, AB of T, and Q_1 is the point common to the perpendiculars drawn through A, B, C to the sides B_1C_1, C_1A_1, A_1B_1 of T_1. Let us designate by B' and C' the intersections of CA and C_1A_1, AB and A_1B_1, which are on d, and write

$$\overrightarrow{A_1P} = k_a\overrightarrow{A_1A}, \qquad \overrightarrow{B_1P} = k_b\overrightarrow{B_1B}, \qquad \overrightarrow{C_1P} = k_c\overrightarrow{C_1C},$$
$$\overline{PA}^2 - \overline{QA}^2 = l_a, \qquad \overline{PB}^2 - \overline{QB}^2 = l_b, \qquad \overline{PC}^2 - \overline{QC}^2 = l_c.$$

If it can be proved that

$$\overline{B'P}^2 - \overline{B'Q}^2 = \overline{C'P}^2 - \overline{C'Q}^2, \tag{1}$$

then it will follow that PQ is perpendicular to d. The fact that

$$B'C/B'A = A_1P \cdot C_1C/A_1A \cdot C_1P = k_a/k_c$$

shows that B' is the centroid of C and A with weighting coefficients k_c and $-k_a$. The theorem of Leibnitz gives

$$\overline{PC}^2 \cdot k_c - \overline{PA}^2 \cdot k_a = \overline{PB'}^2(k_c - k_a) + \overline{B'C}^2 \cdot k_c - \overline{B'A}^2 \cdot k_a$$
$$\overline{QC}^2 \cdot k_c - \overline{QA}^2 \cdot k_a = \overline{QB'}^2(k_c - k_a) + \overline{B'C}^2 \cdot k_c - \overline{B'A}^2 \cdot k_a.$$

Subtracting and dividing by $k_c - k_a$, we have

$$\overline{PB'}^2 - \overline{QB'}^2 = (l_ck_c - l_ak_a)/(k_c - k_a).$$

Relation (1) is equivalent to

$$k_bk_c \cdot (l_b - l_c) + k_ck_a \cdot (l_c - l_a) + k_ak_b \cdot (l_a - l_b) = 0. \tag{2}$$

Since BC is perpendicular to A_1Q,

$$\overline{BA_1}^2 - \overline{CA_1}^2 = \overline{BQ}^2 - \overline{CQ}^2. \tag{3}$$

A_1 is the centroid of A, P with weighting coefficients k_a, -1; hence by the theorem of Leibnitz,

$$\overline{BA}^2 \cdot k_a - \overline{BP}^2 = \overline{BA_1}^2 \cdot (k_a - 1) + \overline{A_1A}^2 \cdot k_a - \overline{A_1P}^2,$$
$$\overline{CA}^2 \cdot k_a - \overline{CP}^2 = \overline{CA_1}^2 \cdot (k_a - 1) + \overline{A_1A}^2 \cdot k_a - \overline{A_1P}^2.$$

By subtraction we obtain, using (3),

$$\overline{BA}^2 - \overline{CA}^2 + \overline{CQ}^2 - \overline{BQ}^2 = (l_a - l_c)/k_a,$$

and by analogy

$$\overline{CB}^2 - \overline{AB}^2 + \overline{AQ}^2 - \overline{CQ}^2 = (l_c - l_a)/k_b$$
$$\overline{AC}^2 - \overline{BC}^2 + \overline{BQ}^2 - \overline{AQ}^2 = (l_a - l_b)/k_c.$$

Addition of the above equations gives us (2) and hence (1), so we conclude that PQ is perpendicular to d. By similar reasoning with the points P and Q_1 we conclude that P, Q, Q_1 are collinear.

COROLLARY. *The conics $ABCPQ_1$ and $A_1B_1C_1PQ$ are equilateral hyperbolas.*

3. Theorem. *If two tetrahedrons $T \equiv ABCD$, $T_1 \equiv A_1B_1C_1D_1$ are bilogic, the center of perspective P and the orthologic centers Q and Q_1 are on a straight line perpendicular to the plane (π) of perspectivity of the two tetrahedrons.*

Proof. Let Q designate the point common to the perpendiculars drawn through A_1, B_1, C_1, D_1 to the faces BCD, CDA, DAB, ABC of T, and Q_1 the point common to the perpendiculars drawn through A, B, C, D to the faces $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ of T_1. If B' is the intersection of the edges CA, C_1A_1, and C' is the intersection of the edges AB, A_1B_1, we may write

$$\overrightarrow{A_1P} = k_a\overrightarrow{A_1A}, \qquad \overrightarrow{B_1P} = k_b\overrightarrow{B_1B}, \qquad \overrightarrow{C_1P} = k_c\overrightarrow{C_1C},$$
$$\overline{PA}^2 - \overline{QA}^2 = l_a, \qquad \overline{PB}^2 - QB^2 = l_b, \qquad \overline{PC}^2 - \overline{QC}^2 = l_c.$$

It will be sufficient to prove that

$$\overline{B'P}^2 - \overline{B'Q}^2 = \overline{C'P}^2 - \overline{C'Q}^2$$

to show that PQ is perpendicular to (π). The demonstration is identical with that carried through for Sondat's theorem.

a). *Another proof.* The plane of perspectivity (π) contains the points

$$B' = (AB, A_1B_1), \qquad C' = (AC, A_1C_1), \qquad D' = (AD, A_1D_1).$$

The altitudes h_1, h_2, h_3, h_4 of the tetrahedron $A_1B'C'D'$ are rulings of the same system of an equilateral hyperboloid (H'). The plane ABQ_1B', which is perpendicular to C_1D_1, contains the altitude h_2. Hence the line AQ_1 meets h_2 and likewise h_3, h_4. AQ_1 is then a ruling of the second system of (H'); it meets h_1 also. Thus the plane AA_1Ph_1, which projects the line APA_1 orthogonally on (π), contains the line AQ_1; P, Q_1, Q are in the plane AA_1Ph_1. Likewise the planes projecting BPB_1, CPC_1, DPD_1 have the same property. It follows that P, Q_1, Q are on a line perpendicular to (π).

b). *Remarks.* The pair of tetrahedrons T, T_1 determine a tetrahedron T_2 of vertices $A_2 = (QA_1, Q_1A)$, $B_2 = (QB_1, Q_1B)$, $C_2 = (QC_1, Q_1C)$, $D_2 = (QD_1, Q_1D)$. Q_1 is the perspective center of T and T_2. Their plane of perspectivity is (π), since the perspective triangles Q_1AB, QA_1B_1 show that the points (AB, A_1B_1),

(QA_1, Q_1A), (QB_1, Q_1B) are collinear. T and T_2 are orthologic; Q is the orthologic center of T and P is the orthologic center of T_2, since the intersection B_2C_2 of the planes Q_1BC, QB_1C_1 is perpendicular to plane ADA_1D_1. Hence the line AA_1P is perpendicular to B_2C_2 and likewise to C_2D_2, D_2B_2. Consequently the perpendicular from A to the face $B_2C_2D_2$ of T_2 contains the point P. This system of tetrahedrons has the following properties [4]:

The tetrahedrons T, T_1, T_2 are bilogic in pairs; the corresponding orthologic centers are Q, Q_1, P. The orthologic center of one of these tetrahedrons is the perspective center of the two others. The three perspectivities have the same perspective plane. Any two of the tetrahedrons T, T_1, T_2 determine the third.

c). THEOREM. *The line $\Delta = PQQ_1$ is a ruling of the second system of the hyperboloids (H) and (H_1) of the altitudes of T and T_1.*

Proof. If A'' is the foot of the altitude AA'' of T, QA_1 is parallel to AA''. The figure $AA''QA_1$ is plane and the line PQ meets AA'' as well as the other altitudes of T. Hence PQ is a ruling of the second system of (H). By permuting A and A_1, T and T_1, it follows that PQ is also a ruling of the second system of (H_1).

4. Polar reciprocation with respect to a quadric. The poles A_1, B_1, C_1, D_1 of the faces BCD, CDA, DAB, ABC of a tetrahedron $T \equiv ABCD$ with respect to a quadric (Q) are the vertices of a tetrahedron T_1. T and T_1 are polar reciprocals with respect to (Q).

CHASLES'S THEOREM. *Two polar reciprocal tetrahedrons T and T_1 with respect to a quadric (Q) are hyperbolic* [5].

COROLLARY 1. *The four lines of intersection of the faces of a tetrahedron T with the polar planes of the opposite vertices with respect to a quadric form a hyperbolic set.*

COROLLARY 2. *If the tetrahedrons T and T_1 are homologic, then T and the tetrahedron formed by the polar planes of the vertices of T with respect to a quadric are also homologic.*

If the quadric is a sphere we have the

THEOREM: *Two tetrahedrons T, T_1 conjugate with respect to a sphere (S) are orthologic and hyperbolic.*

This theorem follows from the fact that the vertices A_1, B_1, C_1, D_1 of T_1 are on the axes of the circles BCD, CDA, DAB, ABC, and the point Q is the center S of the sphere (S).

THEOREM. *If the center S of (S) is on the hyperboloid (H) determined by the altitudes of T, and also on the hyperboloid (H_1) determined by the altitudes of T_1,*

then the lines Δ_a, Δ_b, Δ_c, Δ_d of intersection of the planes of faces BCD and $B_1C_1D_1$, CDA and $C_1D_1A_1$, DAB and $D_1A_1B_1$, ABC and $A_1B_1C_1$ are on a hyperbolic paraboloid.

Proof. Through S there pass rulings G and G_1 of the second system of rulings of (H) and (H_1), respectively. G and G_1 coincide since otherwise they would determine a plane (P) passing through S. As G meets the altitude AA'' of T, G and G_1 would be in the plane $SAA'' \equiv SAA_1$, and by the same argument G and G_1 would also be in the planes SBB_1, SCC_1, SDD_1. This contradicts the assumption that T is a non-degenerate tetrahedron. Hence G and G_1 coincide, and (H), (H_1) have a common ruling passing through S. Furthermore G is in the planes SAA_1, SBB_1, SCC_1, SDD_1, and is perpendicular to Δ_a, Δ_b, Δ_c, Δ_d, which belong to a hyperbolic paraboloid whose director plane is perpendicular to G.

If $(S) \equiv (O)$ is the circumsphere of T, Q and Q_1 coincide with O.

THEOREM. *If the tetrahedrons T, T_1 are bilogic, T is involutory** [6].

The line which joins the perspective center to the common orthologic center O is a ruling of the hyperboloids (H), (H_1). O is on (H).

COROLLARY 1. *If T and its tangential tetrahedron t are bilogic, T is isodynamic and conversely* [7].

In this particular case the lines of intersection of the corresponding faces of T and t are coplanar and conversely [8].

COROLLARY 2. *In an isodynamic tetrahedron the second Lemoine point L is on the hyperboloid (H) and conversely.*

L coincides with the perspective center of T and t.

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2. V. Thébault, Bulletin de l'Académie royale de Belgique, 1921, p. 57.
3. J. Neuberg, Mathesis, 1922, p. 163.
4. C. Servais, Bulletin de l'Académie royale de Belgique, 1923, p. 49.
5. H. F. Baker, Principles of Geometry, vol. 3, p. 41, Ex. 7.
6. J. Neuberg, Mémoire sur le tétraèdre, 1884, p. 58.
7. N. A. Court, Modern Pure Solid Geometry, p. 281.
8. N. A. Court, ibid., p. 281 and V. Thébault, this MONTHLY, 1949, p. 637, advanced problem 4368.

* The term *involutory tetrahedron* used in connection with [6] is the translation of *tétraèdre involutif* used by Neuberg to refer to the fact that the polar line of an edge with respect to the circumsphere meets the respectively opposite edge of the tetrahedron. The term has been accepted by French writers in this field.

MATHEMATICAL NOTES

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TWO NEW INEQUALITIES

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If $p_j \geqq 0$ $(j=1, 2, \cdots, n)$, then the familiar inequality between geometric and arithmetic means yields the relation

$$np_1p_2 \cdots p_n \leqq p_1^n + p_2^n + \cdots + p_n^n. \tag{1}$$

Now suppose that $p_1 \geqq p_2 \geqq \cdots \geqq p_n$ and consider the square arrays

$$\begin{array}{llll} p_1 & p_2 \cdots & p_{n-1} & p_n \\ p_2 & p_3 \cdots & p_n & p_1 \\ \cdots & \cdots & \cdots & \cdots \\ p_n & p_1 \cdots & p_{n-2} & p_{n-1} \end{array} \qquad \begin{array}{lll} p_1 & p_2 \cdots & p_n \\ p_1 & p_2 \cdots & p_n \\ \cdots & \cdots & \cdots \\ p_1 & p_2 \cdots & p_n \end{array} \tag{2}$$

If we denote the array on the left by a_{ij} and that on the right by a'_{ij}, then inequality (1) states that

$$\sum_{j=1}^{n} \prod_{i=1}^{n} a_{ij} \leqq \sum_{j=1}^{n} \prod_{i=1}^{n} a'_{ij}. \tag{3}$$

More generally, consider the rectangular array a_{ij} $(i=1, \cdots, m;$ $j=1, \cdots, n)$ of non-negative numbers: $a_{ij} \geqq 0$. Let a'_{ij} denote the array obtained by rearranging each row in non-increasing order: $a'_{i1} \geqq \cdots \geqq a'_{in}$ $(i=1, \cdots, m)$. Then our main result is the

THEOREM:

$$\sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij} \leqq \sum_{j=1}^{n} \prod_{i=1}^{m} a'_{ij} \tag{I}$$

and

$$\prod_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \geqq \prod_{j=1}^{n} \sum_{i=1}^{m} a'_{ij}. \tag{II}$$

Observe that (II) is, in a sense, dual to (I). For the arrays considered in (2), (II) specializes to essentially the same inequality as (1), now taking the form

$$n^n p_1 \cdots p_n \leqq (p_1 + \cdots + p_n)^n. \tag{4}$$

The case $m=2$ of assertion (I) is treated by Hardy, Littlewood, and Polya (*Inequalities*, p. 261, Theorem 368 and footnote a).

Since the order in which the terms occur within a row of the array a_{ij} is given quite arbitrarily, the theorem really makes an assertion about maxima and minima obtainable by arbitrary rearrangements within the rows of the fixed array a'_{ij}. In order to deal with permutations of the elements within the various rows, we introduce a convenient notation. By a permutation function u, defined on the integers $1, 2, \cdots, n$, we shall mean a mapping of the integers $1, 2, \cdots, n$ onto themselves. Thus, ui $(i=1, \cdots, n)$ is one of the integers $1, \cdots, n$ and u simply rearranges the ordered sequence $1, 2, \cdots, n$. In the place of "permutation function" we shall use simply the word "permutation".

The theorem may now be stated in more convenient form.

THEOREM. *Let sequences* $a_1, \cdots, a_m$ *be defined as follows*:

$$a_i: \quad a_{i1} \geqq a_{i2} \geqq \cdots \geqq a_{in} \geqq 0 \qquad \text{for } i = 1, \cdots, m.$$

Let $x_2, x_3, \cdots, x_m$ *be* $m-1$ *permutations on* $1, \cdots, n$. *Let* x *be the ordered set* $(x_2, x_3, \cdots, x_m)$. *Let* e *be the identity permutation. Finally, let*

$$x(a_1, \cdots, a_m) = \sum_{j=1}^{n} a_{1j}a_{2x_2j}a_{3x_3j} \cdots a_{mx_mj}.$$

Then for every choice of x,

I. $$x(a_1, \cdots, a_m) \leqq e(a_1, \cdots, a_m) = \sum_{j=1}^{n} a_{1j}a_{2j} \cdots a_{mj}$$

II. $$\prod_{i=1}^{n} (a_{1i} + a_{2x_2i} + \cdots + a_{mx_mi}) \geqq \prod_{i=1}^{n} (a_{1i} + \cdots + a_{mi}).$$

Proof of I: The theorem is trivially true for $m=1$. The proof will proceed by an induction on m. Assume that the theorem is valid for all sequences of length $\leqq n$ which number $\leqq m-1$. We seek to prove validity for m sequences all of length n.

To Prove I it will suffice to prove

$$d = e(a_1, \cdots, a_m) - x(a_1, \cdots, a_m) \geqq 0.$$

If we let $b_j = a_{2j} \cdots a_{mj} - a_{2x_2j} \cdots a_{mx_mj}$, we may write

$$d = \sum_{j=1}^{n} a_{1j}b_j.$$

We need the following lemma.

LEMMA 1: $T = b_1 + b_2 + \cdots + b_s \geqq 0$ *for* $s = 1, \cdots, n$.

This lemma is immediately true for $s=1$. Let v_p be any permutation on $1, \cdots, s$ such that $v_p j = x_p j$ if $x_p j \leqq s$ for $j=1, \cdots, s$ and for $p=2, \cdots, m$. Then

$$a_{2v_2 j} \cdots a_{mv_m j} \geqq a_{2x_2 j} \cdots a_{mx_m j} \qquad \text{for } j = 1, \cdots, s.$$

We may now write

$$T = \sum_{j=1}^{s} a_{2j} a_{3j} \cdots a_{mj} - \sum_{j=1}^{s} a_{2x_2 j} a_{3x_3 j} \cdots a_{mx_m j}$$

$$\geqq \sum_{j=1}^{s} a_{2j} a_{3j} \cdots a_{mj} - \sum_{j=1}^{s} a_{2v_2 j} a_{3v_3 j} \cdots a_{mv_m j}.$$

As v_p for $p=2, \cdots, m$ is a permutation on $1, \cdots, s$ we may use our induction hypothesis for sequences of length $s \leqq n$ which number $m-1$. This tells us that the right member of the last inequality $\geqq 0$. Hence $T \geqq 0$ and the lemma is proved.

We return now to the proof of I. We have $a_{11}b_1 \geqq 0$, and from Lemma 1, $a_{11}b_1 + a_{12}b_2 \geqq a_{12}(b_1+b_2) \geqq 0$. We use an induction again. Assume for a fixed $1 \leqq r < n$

$$a_{11}b_1 + \cdots + a_{1r}b_r \geqq a_{1r}(b_1 + \cdots + b_r) \geqq 0.$$

$$a_{11}b_1 + \cdots + a_{1r+1}b_{r+1} \geqq a_{1r}(b_1 + \cdots + b_r) + a_{1r+1}b_{r+1}$$
$$\geqq a_{1r+1}(b_1 + \cdots + b_{r+1}) \geqq 0.$$

Hence, $d = a_{11}b_1 + \cdots + a_{1n}b_n \geqq a_{1n}(b_1 + \cdots + b_n) \geqq 0$. This concludes the proof of I.

Part I of the theorem remains valid if a single negative number appears among all the a_{ij}. However, if two or more negative numbers appear when $m \geqq 3$, there are counterexamples to show that the theorem is then false.

Proof of II: The proof makes use of the following simple lemma.

LEMMA 2. *If* $A+B=C+D$ *and* $|A-B| > |C-D|$, *then* $AB < CD$. *For*, $(A+B)^2 - (A-B)^2 < (C+D)^2 - (C-D)^2$.

Let the minimum be attained for the ordered set of permutations $u = (u_2, \cdots, u_m)$. Then

$$P = \min_{x} \left\{ \prod_{i=1}^{n} (a_{1i} + a_{2x_2 i} + \cdots + a_{mx_m i}) \right\} = \prod_{i=1}^{n} (a_{1i} + a_{2u_2 i} + \cdots + a_{mu_m i}).$$

We shall show that u_i may be replaced by e without changing the value of P, for $i=2, \cdots, m$.

Suppose $u_i = e$ for $i=2, \cdots, g-1$, but $u_g \neq e$. We shall show that u_g may be replaced by e without changing P. Once this is shown, then $m-g+1$ changes of this kind will give the result we are trying to prove. Let $u_g i = i$ for $i=1, \cdots,$

$k-1$, but $u_g k=h\neq k$. In fact, $h>k$. Let $u_g w=k$. Now consider the kth and wth factors of P, represented by K and W respectively. Then

$$K = (a_{1k} + a_{2k} + \cdots + a_{g-1,k} + a_{gh} + a_{g+1,u_{g+1}k} + \cdots + a_{m,u_m k}),$$

and

$$W = (a_{1w} + a_{2w} + \cdots + a_{g-1,w} + a_{gk} + a_{g+1,u_{g+1}w} + \cdots + a_{m,u_m w}).$$

For P to be minimum, no interchange of terms in K and W which have the same first subscript can decrease KW. From our lemma, this will be the case when $|K-W|$ is a maximum. There are two possibilities to be considered.

Case A: $a_{gh}=a_{gk}$. Replace u_g by the permutation u_g' defined by $u_g' i=u_g i$ for all i except for $i=k$, w. $u_g' k=k$, and $u_g' w=h$.

Case B: $a_{gh}\neq a_{gk}$. Then from the hypothesis we must have $a_{gk}>a_{gh}$. If $K\geqq W$, then interchanging a_{gk} and a_{gh} will increase the difference $K-W$ and so decrease the product KW and hence, also P. Thus, if P is minimum, we must have $K<W$. Let R stand for the sum of the first $g-1$ terms in K; and T, for the sum of the first $g-1$ terms in W. Let $S=K-R$ and $U=W-T$. Thus, $K=R+S$ and $W=T+U$. As $w>k$, we must have $a_{ik}\geqq a_{iw}$ for $i=1, \cdots, g-1$. Hence, $R\geqq T$, so $S<U$. Let $K'=R+U$ and $W'=T+S$. If $R>T$ then $K'-W'=|(R+U)-(T+S)|>|(R+S)-(T+U)|=|K-W|$. Hence, using K' and W' in place of K and W decreases KW, and so, also P, which is impossible. So we must have in this case $R=T$, and then $K'-W'=|K-W|$. Using K' and W' in place of K and W does not change the value of P but it does replace a_{gh} by a_{gk}. Successive changes of the type indicated in Case A and the permissible type in Case B will replace u_g by e without affecting P. This completes the proof of part II of the theorem.

A CORRECTION

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In the note "A Simplified Technique for a Tschirnhaus Transformation" (this MONTHLY, vol. 58, pp. 483–484) the last paragraph needs to be completely revised. The first sentence of this paragraph should read: "On the other hand, if $B\neq 0$, and η is found to satisfy an equation of degree three but not one of degree two, we must conclude that $f(x)$ is not irreducible." The remainder of the paragraph not only contains factual errors, but, since $B\neq 0$, is now irrelevant. The following example illustrates the point in question. Let $f(x)=x^4-9x^2+x+3$ and $\eta=1+\rho^2$. Then $B\neq 0$, η satisfies the irreducible equation $x^3-12x^2+27x-17=0$, and $f(x)$ is reducible being identical with $(x+3)(x^3-3x^2+1)$.

CLASSROOM NOTES

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A CORRECTION FOR THE TRAPEZOIDAL RULE

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One approach to the problem of numerical integration is to approximate the integrand with an interpolation formula which is then integrated. We use this procedure to supply a correction to the well-known trapezoidal rule.

Consider

$$\int_{x_0}^{x_1} f(x)dx.$$

We require that $f(x)$ and its first three derivatives be continuous in the interval $x_0 \leqq x \leqq x_1$. We use a linear interpolation formula with correction to approximate $f(x)$.* Let

$$f(x) = I + C + R$$

where

$$I = \frac{(x_1 - x)f(x_0) + (x - x_0)f(x_1)}{(x_1 - x_0)}, \tag{1}$$

the linear interpolation formula, and where

$$C = \frac{[f'(x_0) - f'(x_1)]}{2(x_1 - x_0)}(x_1 - x)(x - x_0), \tag{2}$$

the correction.

Now,

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} Idx + \int_{x_0}^{x_1} Cdx + E_1 \tag{3}$$

where

$$E_1 = \int_{x_0}^{x_1} Rdx. \tag{4}$$

When the integration in (3) is carried out, we get the formula

* A new interpolation formula, P. M. Hummel and C. L. Seebeck, Jr., This MONTHLY, vol. 58, No. 6, 1951, pp. 383–389.

$$\int_{x_0}^{x_1} f(x)dx = \frac{1}{2}[f(x_0) + f(x_1)](x_1 - x_0)$$

(5)

$$+ \frac{1}{12}[f'(x_0) - f'(x_1)](x_1 - x_0)^2 + E_1.$$

The first term is the area of a trapezoid and the second term is a correction for it.

To determine E_1, which is the error due to using the first two terms of (5) as a formula, we have

$$E_1 = \int_{x_0}^{x_1} R dx$$

(6)

$$= \int_{x_0}^{x_1} \frac{(x_1 - x)(x - x_0)}{12(x_1 - x_0)}[f'''(\theta_2)(x_1 - x)^2 - f'''(\theta_1)(x - x_0)^2]dx$$

where $x_0 \leqq \theta_1 \leqq x$ and $x \leqq \theta_2 \leqq x_1$. We write (6) in the form

$$E_1 = \frac{1}{12(x_1 - x_0)}\left[\int_{x_0}^{x_1} (x_1 - x)^3(x - x_0)f'''(\theta_2)dx\right.$$

(7)

$$\left. - \int_{x_0}^{x_1} (x_1 - x)(x - x_0)^3 f'''(\theta_1)dx\right].$$

It is clear that $(x_1-x)^3(x-x_0)$ and $(x_1-x)(x-x_0)^3$ do not change sign in the interval $x_0 \leqq x \leqq x_1$. We use the first mean value theorem for integrals and get

$$E_1 \leqq \frac{1}{12(x_1 - x_0)}\left[f'''(\xi_2)\int_{x_0}^{x_1} (x_1 - x)^3(x - x_0)dx\right.$$

$$\left. - f'''(\xi_1)\int_{x_0}^{x_1} (x_1 - x)(x - x_0)^3 dx\right]$$

or

$$E_1 \leqq \frac{(x_1 - x_0)^4}{240}[f'''(\xi_2) - f'''(\xi_1)] \tag{8}$$

with ξ_1 and ξ_2 between x_0 and x_1. If we now let

$$f(x_0) = y_0, \qquad f'(x_0) = y_0', \cdots,$$

we may write the formula in the more compact form

$$\int_{x_0}^{x_1} f(x)dx = \frac{1}{2}[y_0 + y_1](x_1 - x_0) + \frac{1}{12}[y_0' - y_1'](x_1 - x_0)^2 + E_1 \tag{9}$$

where E_1 is given by (8).

Now, suppose we consider

$$\int_{x_0}^{x_n} f(x)dx \tag{10}$$

where the interval (x_0, x_n) is subdivided by the points

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n.$$

We may integrate over each subinterval (x_{i-1}, x_i), $i=1, 2, \cdots, n$ using (9) on the subinterval (x_0, x_1) and similar formulas for each succeeding subinterval. If we add the results of the n integrations, we get

$$\begin{aligned}\int_{x_0}^{x_n} f(x)dx = {} & \frac{1}{2}\left[(y_0 + y_1)(x_1 - x_0) + (y_1 + y_2)(x_2 - x_1) + \cdots\right. \\ & \left. + (y_{n-1} + y_n)(x_n - x_{n-1})\right] \\ & + \frac{1}{12}\left[(y_0' - y_1')(x_1 - x_0)^2 + (y_1' - y_2')(x_2 - x_1)^2 + \cdots\right. \\ & \left. + (y_{n-1}' - y_n')(x_n - x_{n-1})^2\right] + E\end{aligned} \tag{11}$$

where

$$E = \sum_{i=1}^{n} E_i.$$

This formula may be simplified by subdividing the interval (x_0, x_n) into equal subintervals. Let

$$x_i - x_{i-1} = h, \qquad i = 1, 2, \cdots, n$$

and (11) becomes

$$\begin{aligned}\int_{x_0}^{x_n} f(x)dx = {} & \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right] \\ & + \frac{h^2}{12}\left[y_0' - y_n'\right] + E\end{aligned} \tag{12}$$

where now

$$E \leqq \frac{h^4}{240}\left[f'''(\xi_2) - f'''(\xi_1) + f'''(\xi_4) - f'''(\xi_3) + \cdots + f'''(\xi_{2n}) - f'''(\xi_{2n-1})\right] \tag{13}$$

with ξ_1 and ξ_2 between x_0 and x_1, ξ_3 and ξ_4 between x_1 and x_2 $\cdots$. The formula (12) will be recognized as the familiar trapezoidal rule with the second term an added correction. To get a simpler expression for E, we may replace the terms in

(13) in brackets by $n(M-m)$ where M and m are maximum and minimum $|f'''(x)|$ in the interval $x_0 \leqq x \leqq x_n$ respectively. We now have

$$|E| \leqq \frac{nh^4}{240}(M - m). \tag{14}$$

We may use (14) to compute an upper bound of error committed in using the first two terms of (12) as a formula.

We will compare the results of the formula with those given by several other well-known formulas. We evaluate

$$\int_4^{5.2} \log x \, dx$$

which is 1.82784744, correct to the number of places stated. We let $h=0.2$, $n=6$, $f(x)=\log x$ and $f'(x)=1/x$. We have

$$\begin{aligned} y_0 &= 1.38629436 \qquad y_0' = 0.25 \\ y_1 &= 1.43508453 \\ y_2 &= 1.48160454 \\ y_3 &= 1.52605630 \\ y_4 &= 1.56861592 \\ y_5 &= 1.60943791 \\ y_6 &= 1.64865863 \qquad y_6' = 0.192307692. \end{aligned}$$

The results of this and other formulas appear listed below with their differences from the correct value:

	Result	Error
Trapezoidal	1.82765514	19230×10^{-8}
Simpson's Rule	1.82784726	18×10^{-8}
Weddle's Rule	1.82784741	3×10^{-8}
Trapezoidal rule with correction	1.82784745	1×10^{-8}

Another example is of interest. We evaluate

$$\int_0^1 e^{-x^2} dx,$$

an integral which cannot be obtained in closed form. We let $h=0.2$ and $n=5$ and get

$$\int_0^1 e^{-x^2} dx = 0.744613$$

where $|E| \leqq 0.00005$, using (14).

Certain advantages of the formula should be noted:

1. The formula is simple, requiring that the derivatives be evaluated only at the end points if equal subintervals are chosen.

2. There are no restrictions on the number of subintervals which may be used, whereas Simpson's rule requires an even number and Weddle's rule a multiple of six.

3. Increased accuracy.

VOLUMES OF REVOLUTION

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Assume the parametric equations of a curve to be

$$x = x(t), \qquad y = y(t) \tag{1}$$

where $x(t)$ and $y(t)$ are single-valued, differentiable functions of t for

$$t_0 \leqq t \leqq t_1 \tag{2}$$

such that

$$x(t_0) = x(t_1), \qquad y(t_0) = y(t_1), \tag{3}$$

where t_1 is the least value of $t > t_0$ for which (3) holds.

Let G be the graph of that portion of (1) which satisfies (2) and (3).

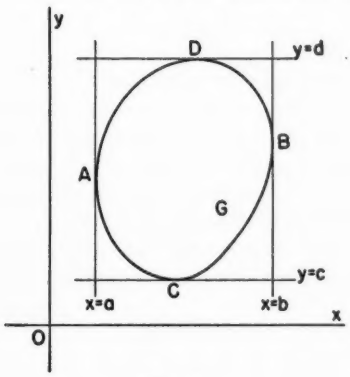

FIG. 1

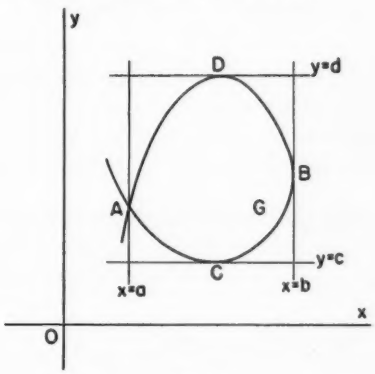

FIG. 2

Graphically, G is a continuous closed curve which does not cut itself. It may be the entire curve (1) (see Figure 1) or a closed portion of (1) (see Figure 2).

Assume that as t increases from t_0 to t_1, the point $P(x, y)$ traces G in the positive direction, where the positive direction is defined as that in which a person walking around the curve has the enclosed area on his left. Let us first

examine the case in which a line parallel to a coordinate axis intersects G in no more than two distinct points.

Let G be within the rectangle determined by the lines $x=a$, $x=b$, $y=c$, $y=d$ (Figures 1, 2), where each of these lines is tangent to G, or passes through a multiple point of (1) but not cutting G in a second distinct point. That is, the lines are through the maximal and minimal values of $x(t)$ and of $y(t)$ in the interval (2).

On arc ACB we may take $y=f_1(x)$ and on arc ADB, $y=f_2(x)$, where $f_1(x)$ and $f_2(x)$ are single-valued, continuous functions of x.

If the x-axis does not cross G, then the volume generated by revolving the area enclosed by G about the x-axis is given by the formula

$$
\begin{aligned}
V &= \pi \int_a^b f_2^2(x)dx - \pi \int_a^b f_1^2(x)dx \\
&= -\pi \int_b^a f_2^2(x)dx - \pi \int_a^b f_1^2(x)dx \qquad (4) \\
&= -\pi \int_G f^2(x)dx.
\end{aligned}
$$

In the last integral, $f(x)=f_1(x)$ if $P(x, y)$ is on arc ACB and $f(x)=f_2(x)$ if $P(x, y)$ is on arc ADB. The letter G at the lower right of the integral sign means that the integral is to be taken around G in the positive direction.

By hypothesis, for points on G

$$
\begin{aligned}
y &= f(x) = y(t) \\
x &= x(t)
\end{aligned}
$$

whence

$$dx = x'(t)dt.$$

We may then write the last integral in terms of t and dt as

$$V = -\pi \int_{t_0}^{t_1} y^2(t)x'(t)dt. \qquad (5)$$

We may also write

$$
\begin{aligned}
V &= 2\pi \int_c^d g_2(y)ydy - 2\pi \int_c^d g_1(y)ydy \\
&= 2\pi \int_c^d g_2(y)ydy + 2\pi \int_d^c g_1(y)ydy \\
&= 2\pi \int_G g(y)ydy
\end{aligned}
$$

or, in terms of t and dt

$$V = 2\pi \int_{t_0}^{t_1} x(t)y(t)y'(t)dt.^* \tag{6}$$

Similar formulas can be written for rotation about Oy.

If as t increases from t_0 to t_1, the point $P(x, y)$ traces G in the negative direction, the above integrals give the negative of V.

The restriction that no line parallel to a coordinate axis cuts G in more than two distinct points insures that G can be separated into two arcs each defined by a single-valued function. This restriction can be removed if, by drawing a finite number of lines, the area enclosed by G can be divided into non-overlapping sub-areas whose boundaries satisfy the previous conditions.

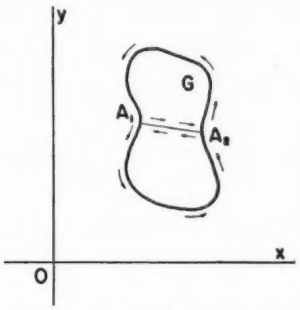

FIG. 3

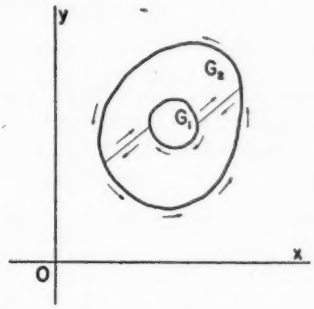

FIG. 4

Figure 3 suggests the method. The integrals taken along A_1A_2 are taken in opposite directions and hence add to zero. The sum of the integrals taken (in the positive direction) around the boundary of each sub-area gives the volume generated by G, and hence the integral taken around G (in a positive direction) gives the volume generated by G.

In Figure 4, the area is bounded by two curves G_1, G_2. By drawing lines as indicated in the figure, the region is divided into sub-areas satisfying the previous conditions. Integrals may be taken around boundaries of sub-areas as indicated.

Calculus texts frequently imply that such volumes must be expressed as the sum of two or more integrals. This paper shows that it is unnecessary to do this, if the boundaries meet the condition given above.

The theory may be applied to regions bounded by suitable arcs of two or more distinct curves. The integrals (5) and (6) in such cases are to be replaced by a sum of integrals, one for each arc, with limits chosen so that P describes the entire boundary in the positive direction. In particular, if the axis of revolution forms a part of the boundary, the limits may be chosen so that P describes

* It may be noted that formula (6) can be derived from (5) by integration by parts.

(in the positive direction) that portion of the boundary which is distinct from the axis of revolution.

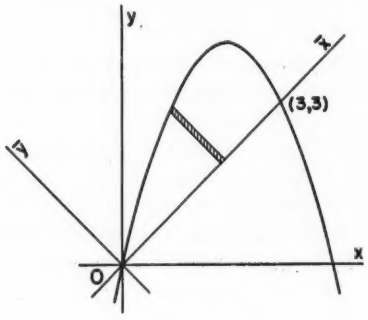

FIG. 5

EXAMPLE: Find the volume generated by rotating about $y=x$ the area bounded by $y=4x-x^2$ and $y=x$.

Referring $y=4x-x^2$ to $y=x$ and $y=-x$ as axes (which we will call $\bar{x}$, $\bar{y}$ axes) we have

$$\bar{x} = \frac{1}{\sqrt{2}}(5x - x^2)$$

$$\bar{y} = \frac{1}{\sqrt{2}}(3x - x^2)$$

as parametric equations of $y=4x-x^2$ in $\bar{x}$, $\bar{y}$ coordinates with x as parameter.

Employing formula (5), and noting that

$$y(t) = \bar{y} = \frac{1}{\sqrt{2}}(3x - x^2)$$

$$x'(t)dt = d\bar{x} = \frac{1}{\sqrt{2}}(5 - 2x)dx$$

we have

$$V = -\pi \int_3^0 \left[\frac{1}{\sqrt{2}}(3x - x^2)\right]^2 \frac{1}{\sqrt{2}}(5 - 2x)dx = \frac{81\sqrt{2}\pi}{20}.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, N. Y. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 996. *Proposed by Josef Andersson, Vaxholm, Sweden*

If a, b, c and r, r_a, r_b, r_c denote the measures of the sides and of the radii of the inscribed and escribed circles of a triangle, show that

$$abc/r \geq a^3/r_a + b^3/r_b + c^3/r_c.$$

E 997. *Proposed by David Ellis, University of Florida*

Let x, y, z, n be positive integers such that $n>1$ and $x^n+y^n=z^n$. Show that

$$z^{n-1} < \min(x^n, y^n) < \max(x^n, y^n) < z^n, \tag{1}$$

$$1/z < \min((x/y)^n, (y/x)^n) < \max((x/y)^n, (y/x)^n) < z. \tag{2}$$

E 998. *Proposed by Leo Moser, Texas Technological College*

Show that the largest area A that can be enclosed by two fixed lines, inclined at an angle $\theta \leqq \pi$, and n straight line segments of unit length is given by

$$A = (n/4)\cot(\theta/2n).$$

E 999. *Proposed by L. G. Johnson, General Motors Corporation, Detroit*

Derive general expressions for the coordinates of the center and for the lengths of the semi-axes of the ellipse formed by the plane $Ax+By+Cz+D=0$ cutting the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

E 1000. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB of a triangle ABC, A', B', C' the feet of the corresponding altitudes, and G the centroid of the triangle. Show that the centroid of triangle $A'B'C'$ and the reflections A_2, B_2, C_2 of the centroids of triangles $AB'C'$, $BC'A'$, $CA'B'$ in the lines B_1C_1, C_1A_1, A_1B_1 are concyclic on a circle with center at G, and that triangles ABC and $A_2B_2C_2$ are inversely similar.

SOLUTIONS

A Variation of the Snow Plow Problem

E 963 [1951, 260]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

It had started snowing before noon and three snow plows set out at noon, 1 o'clock, and 2 o'clock, respectively, along the same path. If at some later time they all came together simultaneously, find the time of meeting and also the time it started snowing.

Solution by L. A. Ringenberg, Eastern Illinois State College. Let n denote the number of hours before noon that it started to snow and let x, y, z denote the distances that the first, second, and third snow plow traveled by t hours past noon. Assume that it snows at a constant rate and that each plow removes snow at the same constant rate. Then, if length units are properly chosen, we have

$$dt/dx = t + n, \qquad x = 0 \quad \text{when} \quad t = 0,$$

with solution

$$t = e^x n - n. \tag{1}$$

Also

$$dt/dy = t - (e^y n - n), \qquad y = 0 \quad \text{when} \quad t = 1,$$

with solution

$$t = e^y(n + 1 - ny) - n. \tag{2}$$

Also

$$dt/dz = t - [e^z(n + 1 - nz) - n], \qquad z = 0 \quad \text{when} \quad t = 2,$$

with solution

$$t = e^z(n + 2 - nz - z + nz^2/2) - n. \tag{3}$$

Let d denote the common value of x, y, and z when the plows meet and let T denote the value of t when the meeting occurs. It follows from (1), (2), and (3) that

$$(T + n)/e^d = n = n + 1 - nd = n + 2 - nd - d + nd^2/2.$$

Solving we get $n = 1/2$ and $T = 3.195$. Therefore it began to snow at 11:30 and the plows met at about 3:12.

Also solved by Peter Bender, J. H. Braun, W. D. Serbyn, Peter Treuenfels, R. E. Wild, and the proposer.

Star-Polygons

E 967 [1951, 338]. *Proposed by L. A. Ringenberg, Eastern Illinois State College.*

Let $P_1, P_2, \cdots, P_n (n>4)$ be n points equally spaced and arranged in order of subscripts on a circle of radius 1, and let S_n be the area of the star formed by the line segments $P_1P_3, P_2P_4, \cdots, P_{n-1}P_1, P_nP_2$. (a) Find a formula for S_n. (b) Find the smallest positive integer n such that $nS_n > (n-1)\pi$. (c) Show that if k is any positive integer greater than 1, then $n^kS_n < (n^k-1)\pi$ for $n=5, 6, 7, \cdots$.

Solution by J. H. Braun, Illinois Institute of Technology. (a) The area of the regular polygon of n sides and radius 1 is $(n/2) \sin (2\pi/n)$. To obtain the area of the star we must subtract the combined area of the isosceles triangles (one in each sector of the polygon), each triangle having vertex angle $(n-2)\pi/n$ and base $2 \sin (\pi/n)$. The area of each isosceles triangle is $\sin^2(\pi/n) \tan (\pi/n)$, whence, by simple trigonometry, we find

$$S_n = n \tan (\pi/n) \cos (2\pi/n). \tag{1}$$

(b) Expanding the expression for S_n we find

$$S_n = \pi - 5\pi^3/3n^2 + \epsilon, \qquad 2\pi^5/15n^4 > \epsilon > 0, \qquad n > 4. \tag{2}$$

Let N be the required minimal integer. Then

$$\pi - 5\pi^3/3N^2 + \epsilon > \pi - \pi/N,$$

or

$$5\pi^2/3N - \epsilon' < 1, \qquad \epsilon' = N\epsilon/\pi.$$

Disregarding ϵ' for the moment we find $N=17$. From (1) we find

$$S_{16} \doteq 2.9296 \not> (1 - 1/16)\pi \doteq 2.9452,$$

whence ϵ' is not sufficiently large to invalidate the solution $N=17$.

(c) From (2) it follows that

$$S_n < \pi[1 - 5\pi^2/3n^2].$$

But

$$\pi[1 - 5\pi^2/3n^2] < \pi(1 - 1/n^2) \leqq \pi(1 - 1/n^k), \qquad k \geqq 2, n > 4.$$

Therefore

$$n^kS_n < (n^k - 1)\pi, \qquad k \geqq 2, n > 4.$$

Also solved by George Millman, Margaret Olmsted, J. A. Tierney, and the proposer.

A Definite Integral

E 968 [1951, 338]. *Proposed by R. M. Redheffer, University of California.*

If $f(s)=\int_{-\pi}^{\pi} e^{\cos x}\cos(\sin x)\cos sx\,dx$, find $\sum_1^\infty f(n)$.

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute. Setting $z=e^{ix}$ in the Maclaurin expansion for e^z we obtain

$$e^{\cos x+i\sin x}=\sum_{n=0}^{\infty}(\cos nx+i\sin nx)/n!.$$

Since the series is uniformly convergent for all x we may multiply both sides by $\cos sx$ and integrate from $-\pi$ to π, obtaining

$$\int_{-\pi}^{\pi} e^{\cos x+i\sin x}\cos sx\,dx=\pi/s!, \qquad s=1,2,\cdots.$$

Taking the real part of the integral we find $f(s)=\pi/s!$, whence

$$\sum_{n=1}^{\infty} f(n)=\pi(e-1).$$

Also solved by F. Bagemihl and W. Seidel, W. Fulks, G. R. Johnston, P. G. Kirmser, M. S. Klamkin, A. E. Livingston, Gordon Raisbeck, Edgar Reich, L. A. Ringenberg, W. D. Serbyn, O. E. Stanaitis, R. E. Wild, and the proposer.

Klamkin called attention to Bierans de Haan Table No. 277.

On the Infinitude of Twin Primes

E 969 [1951, 338]. *Proposed by Solomon Golomb, The Johns Hopkins University.*

Show that a necessary and sufficient condition that there be infinitely many twin primes is that there be infinitely many numbers n not of the form $6ab\pm a\pm b$.

Solution by Azriel Rosenfeld, Columbia University. If p and $p+2$ are primes, then, with the single exception of $p=3$, we must have $p=6n-1$ and $p+2=6n+1$. On the other hand, if a number of either of the forms $6n-1$ and $6n+1$ is composite, then each factor is of one or other of these forms, and we have

$$6n\pm 1=(6a\pm 1)(6b\pm 1)=6(6ab\pm a\pm b)\pm 1.$$

Thus $n=6ab\pm a\pm b$ if and only if either $6n-1$ or $6n+1$ is composite. The desired result now follows.

Also solved by the proposer.

A Parabola Inscribed in a Triangle

E 970 [1951, 338]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that in triangle ABC side BC bisects the segments intercepted by the

other two sides on the tangents to the inscribed parabola having for focus the point A_1 where the symmedian AA_1 intersects the circumcircle.

Solution by O. J. Ramler, Catholic University, Washington, D. C. Since the focus lies on a symmedian, the second focus must, by a fundamental property of foci of conics inscribed in a triangle, lie on the corresponding median. Since the conic is a parabola, the second focus lies at infinity in the direction of the median, and the median is therefore a diameter of the parabola. Let the sides AB and AC of triangle ABC circumscribing the parabola touch the curve in C' and B' respectively. Then the diameter of the parabola bisects $B'C'$, and $B'C'$ is parallel to the tangent to the curve at the point O where the diameter cuts the curve. The tangent BC to the parabola at O is therefore not only parallel to $B'C'$ but is bisected at O. The ideal line in the plane of ABC is tangent to the parabola. By an elementary theorem in the projective theory of conics any tangent cuts four fixed tangents of a conic in a range of points of constant cross ratio. Let any tangent of the parabola inscribed in the triangle ABC meet BC, CA, AB in L, M, N, respectively. Then the cross ratio $(NMLI)$, where I is the ideal point on the tangent, is constant. In particular, when NM assumes the position of BC, M will be at C, N at B, and L at O. But O is the midpoint of BC, whence the constant cross ratio $(NMLI)$ is -1, and L is the midpoint of NM.

Also solved by Joseph Langr.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4468. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be convergent for $|z| < 1$, and further let $0 < |f(z)| \leq 1$ for $|z| < 1$. Prove $|f'(0)| \leq 2/e$.

4469. *Proposed by A. J. Coleman, University of Toronto*

Evaluate the integral

$$\int_0^\infty \frac{e^{-a^2x^2-b^2/x}}{x^2(1+x^2)}\,dx.$$

The dominant term when a is very much smaller than b is also desired. (The integral was met in a discussion of the diffusion of gamma rays.)

4470. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, three lines a', b', c' drawn through the vertices A, B, C determine by their intersections a triangle $A'B'C'$, and their isogonals a'', b'', c'' determine a triangle $A''B''C''$. (1) Show that the orthic triangles of $A'B'C'$ and $A''B''C''$ have equal perimeters. (2) If, further, a', b', c' are equally inclined to BC, CA, AB, show that the circles $A'B'C'$ and $A''B''C''$ are symmetric with respect to the line joining the symmedian point to the circumcenter.

4471. *Proposed by Ky Fan, University of Notre Dame*

Let $K(x, y)$ be a non-negative Lebesgue integrable function over the square $a \leqq x \leqq b$, $a \leqq y \leqq b$. Suppose that B is a positive constant such that $\int_a^b K(x, y)dy \leqq B$ for almost all x in $[a, b]$, and also $\int_a^b K(x, y)dx \leqq B$ for almost all y in $[a, b]$. If two finite-valued functions $f(x)$, $g(x)$ are both non-negative and non-increasing in $[a, b]$, prove that

$$\int_a^b \int_a^b K(x, y)f(x)g(y)dxdy \leqq B\int_a^b f(x)g(x)dx. \tag{1}$$

[For the particular case when $K(x, y)$ is constant, (1) is known as Tchebychef's inequality. See Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934, p. 168, Theorem 236.]

4472. *Proposed by G. T. Williams, Elmont, New York*

If n is an odd integer >1, the curve $x^n+y^n=1$ has the line $x+y=0$ as asymptote, and the area in the second (or fourth) quadrant between the curve and its asymptote exists and is equal to the area in the first quadrant divided by $2\cos(\pi/n)$.

SOLUTIONS

Linear Combination of Vectors with Non-negative Coefficients

4395 [1950, 342]. *Proposed by David Gale, Princeton University*

Let S be a set of vectors (points) in n-space with the property that every vector in n-space can be expressed as a linear combination of vectors from S with non-negative coefficients. Show that one can find a subset R of S containing not more than $2n$ elements such that R has this same property.

Solution by the Proposer. Since S generates all of n-space it must contain some set of n linearly independent vectors, $v_1, \cdots, v_n$. We first observe that if we let $v_0 = -\sum_{i=1}^n v_i$, then every vector can be expressed as a non-negative

linear combination of the vectors $v_0, v_1, \cdots, v_n$. For let v be any vector in n-space. Then

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

If all α_i are non-negative there is nothing more to prove. If not, suppose α_1 is the smallest of the α_i. Then

$$v = \sum_{i=1}^{n} (\alpha_i - \alpha_1)v_i + (-\alpha_1)v_0$$

where again all coefficients are non-negative. To prove the desired result then we need only show that v_0 can be expressed as a non-negative linear combination of not more than n vectors from S. Now by assumption

$$v_0 = \sum_{i=1}^{k} \lambda_i u_i, \qquad \lambda_i \geqq 0,\ u_i \in S. \tag{1}$$

If $k \leqq n$ we are finished. If not, we can find n vectors from among the u_i, say the first n, such that

$$v_0 = \sum_{i=1}^{n} \beta_i u_i \tag{2}$$

and again we only worry about the case where some β_i is negative. Then let $\theta = \max(-\beta_i/\lambda_i)$. Then θ is positive and suppose it is $-\beta_1/\lambda_1$. Then from (1) and (2) we get

$$(1 + \theta)v_0 = \sum_{i=0}^{k} \gamma_i v_i, \qquad \gamma_i = \theta\lambda_i + \beta_i \geqq 0.$$

But $\gamma_1 = 0$, so that after dividing by $(1+\theta)$ we have expressed v_0 non-negatively in terms of $k-1$ vectors from S. By repetition of this argument, the result follows.

Also solved by L. M. Blumenthal and J. W. Gaddum (jointly), V. L. Klee, Jr., A. E. Taylor, and Albert Wilansky.

Editorial Note. Klee and Taylor point out the equivalence of the present result with the following known theorem: ***Any point interior to the convex hull of a set E in an n-dimensional euclidean space is interior to the convex hull of some subset of E containing at most $2n$ points.*** See Bull. Amer. Math. Soc., 53 (1947), 299–301.

The Lemoine Point of a Conic for an Inscribed Triangle

4398 [1950, 343]. *Proposed by Lucien Droussent, Laon, France*

The Lemoine point of a rectangular hyperbola for an inscribed triangle T

lies on the orthic axis of T, and the polar of this point with respect to the hyperbola is tangent to the conic which touches the sides of T at the feet of its altitudes and has the symmedian point of T for center. (The tangents of a conic at the vertices of an inscribed triangle T form a triangle which is perspective to T. The center of perspectivity is called the *Lemoine point* of the conic for the triangle T).

Solution by E. D. Camier, Berkenhead, Cheshire, England. If the triangle T, with vertices A, B, C, is taken as reference triangle in a system of trilinear coördinates, the equilateral hyperbola has the equation $\sum lyz=0$ where $\sum l \cos A=0$. The Lemoine point for T is then (l, m, n), so that the previous relation means that (l, m, n) lies on the orthic axis whose equation is $\sum x \cos A=0$.

The tangential equation of the conic inscribed in T and having (p, q, r) as center is $\sum a(-ap+bq+cr)\mu\nu=0$. Hence when the symmedian point (a, b, c) is the center, the equation is

$$\sum \mu\nu \cos A = 0. \tag{1}$$

The polar of (l, m, n) with respect to $\sum lyz=0$ is $\sum x/l=0$ and the condition that this line should touch the conic (1) is $\sum l \cos A=0$, which is satisfied by hypothesis. Finally, the conic (1) touches BC at $(0, \cos C, \cos B)$, which is the foot of the perpendicular from A to BC.

This result can be generalized. If the Lemoine point of a conic Γ for an inscribed triangle T lies on a fixed straight line L, the polar of this point with respect to Γ touches a second conic Γ', where Γ' is inscribed in T and has its center at the pole of L with respect to the conic which touches T at the midpoints of its sides.

Also solved by the Proposer.

Sequences of Greatest Integer Functions

4399 [1950, 343]. *Proposed by Ky Fan, University of Notre Dame*

Let $f(n)$ and $g(n)$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1)=1$.

(2) $g(n)=na-1-f(n)$, a being an integer >4.

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1)$, $f(2), \cdots, f(n); g(1), g(2), \cdots, g(n)$.

Prove that there exist constants α and β such that

$$f(n) = [\alpha n], \qquad g(n) = [\beta n].$$

The brackets denote, as usual, the greatest integer function.

Solution by H. S. Zuckerman, University of Washington, Seattle. Let α, β be the roots of $x^2-ax+a=0$. Since $a>4$, α, β are real and we take $\alpha<\beta$. We then have $\alpha+\beta=a$, $\alpha\beta=a$, $1/\alpha+1/\beta=1$, $1<\alpha\leqq 2$, $2\leqq\beta$. Further α, β are irrational

since if one were rational both would be integers and we would have $\alpha=2, \beta=2, a=4$. We have immediately

$$[\alpha 1] = 1. \tag{1}$$

Furthermore, for $n \geqq 1$, we have

$$[\beta n] = [(a - \alpha)n] = na - 1 - [\alpha n]. \tag{2}$$

Now if $[\alpha n] = [\beta m] = k$ with n, m positive integers, then

$$\alpha n = k + \theta, \qquad \beta m = k + \phi, \qquad 0 < \theta < 1, 0 < \phi < 1,$$

and hence

$$n + m = k\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \frac{\theta}{\alpha} + \frac{\phi}{\beta} = k + \frac{\theta}{\alpha} + \frac{\phi}{\beta}$$

which is impossible since $0<\theta/\alpha+\phi/\beta<1$. We therefore have

$$[\alpha n] \neq [\beta m] \qquad \text{for any integers } n, m > 0. \tag{3.1}$$

We have also

$$[\alpha(n + 1)] \geqq [\alpha n] + 1, \qquad [\beta(n + 1)] \geqq [\beta n] + 2 > [\alpha n] + 1. \tag{3.2}$$

Finally let $k>0$ be any integer and let $n=[(k+1)/\alpha]$. If $n>k/\alpha$ then $k<\alpha n <\alpha(k+1)/\alpha=k+1$ and $[\alpha n]=k$. If $n<k/\alpha$ then

$$\beta(k - n) > \beta k - \frac{\beta}{\alpha} k = \beta k\left(1 - \frac{1}{\alpha}\right) = k,$$

$$\beta(k - n) < \beta k - \beta\left(\frac{k + 1}{\alpha} - 1\right) = k + 1,$$

from which we have $[\beta(k-n)]=k$. This with (3.1) and (3.2) shows that

(3.3) each $k = 1, 2, 3, \cdots$ appears once and only once in one or other of the two sequences $[\alpha n]$, $[\beta n]$.

Now (3.3) with (3.2) proves that

(3) $[\alpha(n + 1)]$ = the least natural number distinct from $[\alpha 1], [\alpha 2], \cdots, [\alpha n]; [\beta 1], [\beta 2], \cdots, [\beta n]$.

Formulas (1), (2) and (3) are just the definition of $f(n)$, $g(n)$ and, since the sequences so defined are unique, we have $f(n) = [\alpha n]$, $g(n) = [\beta n]$.

We remark that the only part of the proof which requires a to be an integer is (2). Therefore all the other results are valid for any irrational α, β with $1/\alpha +1/\beta=1$. This result is problem 9, p. 98 in *Elementary Number Theory*, Uspensky and Heaslet.

Also solved by the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Theory of Probability. By M. E. Munroe, McGraw-Hill Book Company, Inc., New York. 1951. 8+213 pages. $4.50.

The problem of presenting probability to mathematically immature undergraduates is a difficult one because measure theory, which is essential to a unified treatment, has not yet filtered down to the undergraduate curriculum. One way out of the difficulty is to confine attention to the discrete case in order to drive home fundamental ideas and modern methods in a simple context. (Examples are the recent texts by J. Neyman and by William Feller.) The more usual solution is to treat the discrete and continuous cases within the limits of the students' knowledge of calculus. Professor Munroe's book brings some welcome innovations to this approach. Probability is defined axiomatically. Discrete and continuous variables are treated side by side in order to bring out the analogy and even to suggest explicitly the general unified theory. Almost a third of the book is devoted to limit theorems in order to give the student "the true flavor of modern probability theory." Topics involving mathematics beyond elementary calculus are treated intuitively with frank acknowledgment of lack of rigor. The book is suitable for a one semester course whose purpose is to impart a speaking acquaintance with significant notions, techniques, and results of modern probability theory.

Space does not permit mentioning minor good and bad features, such as the very clear discussion of the chi-square distribution and the regrettable listing of "suggested procedures" as numbered theorems, but attention should be called to the weakness of the discussion of the meaning of probability preceding the axiomatic definition. The notion of "likelihood" is used without any indication of its meaning except that it appears to be an entity of which probability is the measure. Logical certainty is confused with unit probability on page 10, only to be correctly distinguished from it nine pages later. The notion of frequency is introduced, but probability is identified with "expected" frequency. It would be better to state explicitly the contending views on the foundations of probability instead of giving an eclectic presentation which leaves the student unaware of the existence of any consistent approach and unprepared to cope with the conceptual difficulties that arise in statistics.

KENNETH MAY
Carleton College

Ordinary Differential Equations. By Michael Golomb and Merrill Shanks. New York, McGraw-Hill, 1950. ix+356 pp. $3.75.

Differential Equations. Revised Third Edition. By H. B. Phillips. New York,

John Wiley and Sons, Inc., 1951. viii+149 pp. $3.00.

Differential Equations. By J. E. Powell and C. P. Wells. Boston, Ginn and Company, 1950. vi+205 pp. $3.00.

These three textbooks are designed for use in undergraduate courses on ordinary differential equations as offered at the high sophomore or low junior level of almost every institution that has an engineering or science major for a bachelors degree. The present brief review is not intended as a comparison of the books but rather as an indication of the features of each.

The books by Phillips and Powell-Wells are somewhat brief and would be fully used in a five quarter-hour or three semester-hour course. Either of these would seem suitable for a course that requires no more prerequisite than the usual first course in differential and integral caculus. The book by Golomb and Shanks contains sufficient material for more than a one semester course but selected omissions readily adapt it to such courses. This latter book includes a summary of the background material, above the elementary calculus level, needed for the text. The reviewer is of the opinion that best results would be obtained from use of this text in courses that are preceded by some advanced calculus.

Golomb and Shanks have written a book that bridges a serious gap in the textbook literature on differential equations. An institution that can include a course at the level of this book will prepare its students for further work with differential equations, or applications of such equations, in a much more substantial way than can be achieved through the usual, problem-solving courses now given in many colleges. The entire book is written with care for mathematical accuracy in statements of theory back of each operation presented. Much thought and ingenuity have been successfully used in bringing correct definitions, concepts, existence theorems, and procedures within reach of undergraduate students of reasonable ability. There are many sections of the book that merit commendation but space limitations on this review do not permit their enumeration. Among these are: the distinction between "a" and "the" when used in connection with "general solution" (page 23); the discussion of $y'=2\sqrt{y}$ on page 29; the caution on page 53 about use of the vague formula (showing integral signs without limits and using x ambiguously) frequently given for "solving" a first-order, linear equation; and the well-designed approach to Green's functions given on pages 259–270.

The Powell-Wells book is written with primary view to the needs of engineering students. It features a grouping of solution methods under such general headings as: (a) variables separable or reducible to variables separable; (b) exact equations or those reducible to exact equations by means of integrating factors; (c) solution by infinite series. Through such groupings the authors hope to correct an impression on the part of students that "a course in differential equations is nothing but a study of numerous and rather disconnected methods of solving very special types of equations." Two full chapters are devoted to applications of differential equations. Problems are of *practice*, *general*,

or *miscellaneous* type according as they illustrate the principles just preceding the exercise, require the student to seek a method of solution from the full body of preceding text, or include extensions beyond anything given in the text. The reviewer seriously questions the advisability of having the *practice* problems embody the principle which the authors state as follows: "In these lists we have included problems that cannot be worked by the methods of that section and often not by any method previously discussed. It is hoped that this plan will emphasize the importance of knowing where as well as how to apply a certain method." It is likely that an encounter with a problem of the type described in this quote would cause serious loss of confidence by students of the maturity level expected in a course from the text. The authors cite judgment factors which lead to omission of certain points of strict mathematical rigor. The reviewer is not unsympathetic to this view but feels that some opportunities were missed where, with no loss of teaching quality, more precise formulas and language could have been used. For example, limits might have been used on the integral signs that appear in many of the formulas given.

The book by Phillips is a third edition of a text that has had wide use since its first edition appeared in 1922. It is designed for, and well adapted to, the one-quarter or one-semester course that is almost standard in the undergraduate colleges of the country. The present edition contains new problems and several rewritten sections that reflect the experience gained from classroom use of earlier editions. The book has achieved a good balance between applied problems and problems of theoretic nature. Care has been exercised in handling points of mathematical rigor as is evidenced by the inclusion of section 10, page 18, where *integration between limits* is carefully noted.

The typography of the three books under review is uniformly good. The publisher of each is to be congratulated upon the careful work of his staff in the editing and composition of the book.

W. M. WHYBURN
University of North Carolina

NEW BOOKS RECEIVED

Vorlesungen über Differential und Integralrechnung, Zweiter Band, *Differentialrechnung auf dem Gebiete mehrerer Variablen*. By A. Ostrowski. Basel, Verlag Birkhäuser, 482 pp. Swiss fr. 67.

An Introduction to the Theory of Control in Mechanical Engineering. By R. H. MacMillan. Cambridge University Press, 1951. xiv+195 pp. $6.00.

Elementary Problems in Engineering. By H. W. Leach and G. C. Beakley. New York, the Macmillan Co., 1951. 14+269 pp. $3.50.

Mathematische Grundlagen der höheren Geodäsie und Kartographie. By R. König and K. H. Weise. Berlin, Göttingen, Heidelberg; Springer-Verlag; 1951. 18+522 pp. 49.60 German Marks.

Linear Computations. By P. S. Dwyer. New York, John Wiley and Sons, 1951. xi+344 pp. $6.50.

Index Numbers. By B. D. Mudgett. New York, John Wiley and Sons, 1951. x+135 pp. $3.00.

A Concise History of Astronomy. By Peter Doig. New York, Philosophical Library, 1949. xi+320 pp. $4.75.

Everyday Algebra, Intermediate Course. By William Betz and A. P. Windt. Boston, Ginn and Company, 1951. 566 pp. $2.48.

Practical Mathematics, Part IV, Trigonometry and Logarithms. Fifth Edition. By Claude Irwin and S. F. Bibb. New York, McGraw-Hill Book Co., 1951. xii+193 pp. $2.60.

Mathematics Essential for Elementary Statistics. Revised Edition. By Helen M. Walker. New York, Henry Holt and Co., 1951. xiii+382 pp.

Gewöhnliche Differentialgleichungen Sammlung Göschen. By Dr. Guido Hoheisel. Berlin, 1951. 129 pp. +13.

The Mathematics of Finance. By F. C. Smith, New York, Appleton-Century-Crofts, Inc. xii+356 pp.

The Teaching of Mathematics. By D. R. Davis. Cambridge, Addison-Wesley Press, Inc., 1951. xv+415. pp. $4.50.

Integral Transforms in Mathematical Physics (Methuen's Monographs). By C. J. Tranter. New York, John Wiley and Sons, Inc. ix+118 pp. $1.50.

Analytic Geometry and Calculus. By W. R. Longley, P. F. Smith and W. A. Wallace. Boston, Ginn and Company, 1951. ix+578 pp+xx. $5.00.

Machine Shop Mathematics. Second Edition. By Aaron Axelrod. New York, McGraw-Hill. xi+359 pp. $3.60.

Nomographic Charts. By C. A. Kulmann. New York, McGraw-Hill Book Co. xi+244 pp. $6.50.

The Physical Sciences, 3rd ed. By E. J. Cable, R. W. Getchell, and W. H. Kadesch. New York, Prentice-Hall, 1951. xvii+496 pp. $5.50.

Applied Mechanics for Engineers. By Sir Charles Inglis. Cambridge, University Press, 1951. xii+404 pp. $7.50.

College Algebra. Second Edition. By R. W. Brink. New York, Appleton-Century-Crofts, Inc. xvi+495 pp.

Intermediate Algebra. Second Edition. By R. W. Brink. New York, Appleton-Century-Crofts, Inc., 1951. xi+295 pp.

Statistische Methoden. By A. Linder. Switzerland, Verlag Birkhäuser, Basel, 1951. 238 pp. 30 Fr.

Trigonometry for Today. By Milton Brooks and A. C. Schock. New York, Harper and Brothers, 1951. ix+102 pp. $2.96.

Theory of Perfectly Plastic Solids. By William Prager and P. G. Hodge. New York, John Wiley and Sons, Inc., 1951. x+264 pp. $5.50.

Mathematics of Finance. By A. E. May. American Book Company, 1951. viii+264 pp.

Statistical Methodology Reviews. By O. K. Buros. New York, John Wiley and Sons, 1951. 457 pp. $7.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Pi Mu Epsilon, University of Kansas

The *Kansas Alpha* chapter of *Pi Mu Epsilon* held its annual spring banquet in April. The speaker, Prof. E. S. Robinson, gave an address on *Why some philosophers are interested in mathematics.*

The winter initiation and tea was held in January and in addition there were various business meetings during the year.

The officers elected for the year 1951–52 are: Director, Robert Fisher; Vice-Director, David Murcray; Secretary, James Larkin; Treasurer, Kathleen O'Donnell; Corresponding Secretary, Wealthy Babcock.

Mathematics Club, Case Institute of Technology

The *Mathematics Club* at the Case Institute of Technology held six meetings during the year 1950–51, at which the following talks were presented:

The calculus of variations, by J. T. Morse

The concept of order of magnitude in optics, by P. A. Wayman

The mathematical theory of games, by M. D. Morley

The Ritz method in the calculus of variations, by Dr. Robert Clark

Matrices and transformations, by J. T. Morse

The computer at Case for solving simultaneous equations, by C. A. Young.

The Officers were: President, J. T. Morse; Faculty Advisor, Prof. Max Morris.

Mathematics Section, Chamberlin Science Club, Beloit College

The program of the *Mathematics Section* of the *Chamberlin Science Club* of Beloit College for 1950–51 was:

A method for solving certain quartic equations, by Donald Walsh

Duplication of the cube, by Marilyn Marshall

Hyperbolic functions, by Margery Jenkins

Early arithmetic, by Richard Bening

Some interesting problems from Mathematical Snapshots, by Priscilla Kong

Find the digit represented by each letter, by Arthur Ahlgrim

The life of Isaac Newton, by Phyllis Willoughby

The beginnings of numbers, by Fern Lohrber.

The officers for the year 1951–52 are: President, Richard Bening; Vice-President and Program Chairman, Fern Lohrber; Secretary, Marilyn Marshall.

Kappa Mu Epsilon, Baldwin-Wallace College

Talks given before the *Ohio Gamma* chapter of *Kappa Mu Epsilon* during the year 1950–51 were:

Engineering education in England, by Prof. Phillip Clyne of the University of Chelmsford, England

The mathematics of democracy, by Prof. John Wilson

Some applications of the laws of chance, by Jack Hafferkamp

Fun with figures, by Kenneth Buser

Professional opportunities in mathematics, by Glenn Carter

The mathematics of life insurance, by Ellsworth Strock, Assistant Actuary of the Prudential Insurance Co. of America.

The officers elected for the year 1951–52 are: President, Ruth Oberer; Vice-President, Kenneth Buser; Secretary, Carol Schirmer; Treasurer, Donald Larsen; Corresponding Secretary and Faculty Advisor, Prof. Paul Annear.

Mathematics Club, Immaculate Heart College

The following papers were presented at regular meetings of the *Mathematics Club* of Immaculate Heart College of Los Angeles during 1950–51:

Careers for women in mathematics, by Romayne MacFarlane and Mary Lou Wagner

Works and life of Einstein, by Mary Parker and Rita Anerson

Inversion, by Lillian Hampton

History of linkages and their use in describing approximate and exact straight line motion, by Glenna Christie

Linkages which use the principle of inversion, by Rita Ricciardi.

At each meeting of the club a period was devoted to mathematical recreations followed by refreshments and a social hour.

An award for excellence in freshman mathematics was won by Lillian Hampton.

Officers for the year 1950–51 were: President, Mary Lou Wagner; Secretary-Treasurer, Romayne MacFarlane.

Pi Mu Epsilon, Oregon State College

The *Oregon Beta* Chapter of *Pi Mu Epsilon* reports that for the academic year 1950–51 six meetings were held including the annual initiation and banquet. The following papers were presented:

Linear Diophantine equations, by Philip Anselone

Arithmetic of the complex domain, by Robert Brown

Theory of runs, by Gene Thompson

Different sizes of infinities, by Dr. James Price.

An annual Mathematics Contest was inaugurated. Prize winners were: William Gribble, first prize; Richard Lee Adey, second prize; Marshall McMurran, third prize.

Following the untimely death of the second prize winner, the chapter decided to perpetuate his memory by designating the first prize as the *Richard Lee Adey Award.*

The initiation of new members and the annual banquet of the chapter were held in May. Forty-one new members were initiated.

Officers elected for 1951–52 are: Director, Patricia Pearson; Vice-Director, Dallas Banks; Secretary, Robert Brown; Treasurer, Prof. G. A. Williams.

Newtonian Society, Lehigh University

The *Newtonian Society* of Lehigh University is open to Sophomores and Juniors who have demonstrated outstanding ability in mathematics during their freshman year. During the academic year 1950–51, the society held seven meetings. The following programs were presented:

The International Congress of Mathematicians, by Prof. J. O. Chellevold

Magic squares, by Bruce Reinhart

Actuarial mathematics, by Prof. Frank Beale

The game of Nim, by Benjamin Kenny

Going to extremes—Greek style, by Prof. A. A. Wilansky

Elliptic integrals, by Hubert Snyder

A simple branching process, by Dr. S. Goldberg.

Officers elected are: President, Robert Moore; Vice-President, Fred Otter Jr.; Secretary, Donald Clapp; Faculty Advisor, Prof. J. O. Chellevold.

Pi Mu Epsilon, Northwestern University

The program of the *Northwestern University* chapter of *Pi Mu Epsilon* for the year 1950–51 was as follows:

Theorems of the universe, by Dr. W. S. Krogdahl of the Department of Astronomy

Interesting facts about numbers, by Dr. R. P. Boas

Philosophy and mathematics, by Dr. Douglas Morgan of the Department of Philosophy

Science and religion, by Dr. Edson Peck of the Department of Physics

Analogue computer, by Dr. Elliott Buell, as part of a tour through the Technological Institute.

First and second prizes in the annual mathematics contest were awarded to Franklin Peterson and Donald Malm, respectively.

The officers elected for 1951–52 are: President, Richard Goldberg; Vice-President, Franklin Peterson; Treasurer, David Whitehouse; Secretary, Grant Steffen.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

JOURNAL OF THE GRADUATE INSTITUTE FOR APPLIED MATHEMATICS

The Graduate Institute for Applied Mathematics announces the forthcoming publication of *The Journal of Rational Mechanics and Analysis*, edited by T. Y. Thomas and C. A. Truesdell, with the assistance of David Gilbarg, Vaclav Hlavaty, and Eberhard Hopf. The cooperation of twenty-five European and American specialists in mechanics and analysis has been secured. About 600 pages per year will be published, the first number to appear in January, 1952. The printing is done by the Waverly Press. *The Journal of Rational Mechanics and Analysis* nourishes mathematics with physical applications, aiming especially to close up the rift between "pure" and "applied" mathematics and to foster the discipline of mechanics as a deductive, mathematical science in the classical tradition. Its scope comprises those parts of pure mathematics or other theoretical sciences which contribute to mechanics; among the included fields are all branches of analysis, differential geometry, analytical dynamics, elasticity, fluid dynamics, plasticity, thermodynamics, relativity, and statistical mechanics. Engineering applications, numerical work, perturbation procedures, etc., are acceptable only as incidental illustration in a paper devoted to sound mathematical theory. Empirical and semi-empirical conjectures, book reviews, notices, etc., are excluded. Each paper must meet a standard of deductive rigor set by the best work in its field. Contributed papers must contain original research. The editors may occasionally invite an expository paper in a field where they perceive a genuine need.

English, French, German, and Italian are the languages of the Journal. A high expository level is desired, and papers written in an excessively condensed style will not be printed.

It is preferred that an author who desires to publish in the Journal send his manuscript to a member of the Editorial Board, although it is also permissible to send it to the Editors. Names and addresses of the members of the Editorial Board may be found in the Journal.

PERSONAL ITEMS

Mrs. Jewell H. Bushey, chairman of the Department of Mathematics of Hunter College, Dr. Mina S. Rees who is Director of Mathematical Science, Office of Naval Research, and Assistant Professor Edith R. Schneckenburger of the University of Buffalo were appointed to serve as delegates of the Association at a national conference on Women in the Defense Decade which was held in New York City on September 27–28, 1951.

Brother Damian Connelly of LaSalle College represented the Association at

the Centennial Program of St. Joseph's College in Philadelphia, Pennsylvania.

Professor J. S. Frame of Michigan State College was the delegate of the Association at the inauguration of President H. H. Hatcher of the University of Michigan on November 27, 1951.

Professor S. A. Jennings, University of British Columbia, represented the Association at the installation of Chancellor Sherwood Lett of the University of British Columbia on October 26, 1951.

Professor C. G. Latimer of Emory University was the representative of the Association at the inauguration of President W. M. Alston of Agnes Scott College on October 22–23, 1951.

Professor Saunders MacLane of the University of Chicago was the representative of the Association at the inauguration of Chancellor L. A. Kimpton of the University of Chicago on October 18, 1951.

Dr. C. V. Newsom, Associate Commissioner of Higher Education of the State of New York, was appointed to represent the Association at the inauguration of President R. R. De Marco of Finch Junior College, New York City, on November 2, 1951.

Dr. M. M. Slotnick of Humble Oil and Refining Company, Houston, Texas, represented the Association at the inauguration of President R. O. Lanier of Texas Southern University on November 4, 1951.

Professor Charles Wexler, Arizona State College, was appointed to represent the Association at the inauguration of President R. A. Harvill of the University of Arizona on November 16, 1951.

Professor C. R. Wylie, Jr., University of Utah, served as the delegate of the Association at the inauguration of President E. L. Wilkinson of Brigham Young University on October 8, 1951.

At the celebration of the Four Hundredth Anniversary of the National University of Mexico the degree of Doctor Honoris Causa was conferred on Professor Garrett Birkhoff of Harvard University, Professor Solomon Lefschetz of Princeton University and Professor Norbert Wiener, Massachusetts Institute of Technology.

Mr. J. B. Herreshoff IV of the University of California at Berkeley was the recipient of the William Lowell Putnam Scholarship at Harvard University for the Eleventh Annual Competition.

Dr. D. W. Snader, professor of mathematical education at the University of Illinois was honored at the Home Coming Celebration of Albright College with an Alumni Citation. Professor Snader was unable to attend the festivities because he was serving on the Institute for Educational Leadership Program in Hiroshima, Japan.

Beloit College makes the following announcements: Dr. H. H. Conwell has retired from his position as Dean of the College and Chairman of the Department of Mathematics but is continuing to teach in the Department; Professor R. C. Huffer has been promoted to the chairmanship.

Catholic University announces the following: Professor A. E. Landry has

retired with the title of Professor Emeritus; Dr. R. W. Moller has been promoted to an associate professorship.

At Florida State University: Dr. H. E. Taylor has been promoted to an assistant professorship; Dr. Nicholas Heerema, formerly graduate student and research associate at the University of Tennessee, has been appointed to an assistant professorship; Mr. H. H. Barnett, previously a graduate student at the University of Minnesota, has been appointed Acting Asssistant Professor; Miss Mary C. Heath, formerly graduate student at the University of Kentucky, has been appointed temporary instructor.

Johns Hopkins University announces: Assistant Professor F. I. Mautner of Pennsylvania State College has been appointed to an assistant professorship; Dr. J. M. Cook, formerly a graduate student, has been appointed to an instructorship.

Memphis State College reports the appointment of Mr. G. N. Wollan as assistant professor and the appointment of Miss Kathleen Cannon as instructor.

State University of Iowa announces the following: Associate Professor E. N. Oberg has been granted a leave of absence for the year 1951–52 to serve with North American Aviation, Incorporated, Downey, California; Assistant Professor Byron Cosby, Jr., has been promoted to an associate professorship.

At the University of Arizona: Dr. K. A. Fowler, who was an assistant at the University of Michigan, has been appointed to an assistant professorship; Mr. Robert Krans, a graduate student at the University, has been appointed to an assistantship.

The University of Florida makes the following announcements: Professor T. M. Simpson, who has been Head Professor of Mathematics, has retired with the title of Professor Emeritus; Professor F. W. Kokomoor, who has been serving as Professor of Mathematics and Chairman of Freshman Mathematics, has been promoted to the position of Head Professor of Mathematics and is continuing in the position of Chairman of Freshman Mathematics; Associate Professor C. B. Smith has been promoted to a professorship.

The University of Georgia announces that it has purchased the library of the American Mathematical Society. The library contains more than thirteen thousand volumes including many mathematical journals.

The University of Southern California reports the following: Associate Professor D. H. Hyers has been promoted to a professorship; Assistant Professors James Dugundji and A. L. Whiteman have been promoted to associate professorships; Dr. Milton Drandell, previously a teaching assistant at the University of California at Los Angeles, has been appointed Lecturer.

Mr. Walter Abramowitz has been appointed Research Fellow at Polytechnic Institute of Brooklyn.

Graduate Assistant R. P. Bacon of Miami University has accepted a position at Douglas Aircraft Company, El Segundo, California.

Mr. W. G. Bade, who has been a teaching assistant at the University of

California at Los Angeles, has been appointed to an instructorship at the University of California at Berkeley.

Associate Professor Frances E. Baker of Vassar College has been promoted to a professorship.

Mr. E. W. Banhagel, previously an instructor at Northwestern University, has accepted a position as mathematician in the Research Laboratories of Bendix Aviation Company, Detroit, Michigan.

Assistant Professor W. R. Baum of the University of Tennessee has been appointed to an assistant professorship at Syracuse University.

Assistant Professor S. Louise Beasley of Lindenwood College for Women has been promoted to an associate professorship.

Assistant Professor O. B. Bennett of Washington College has been promoted to an associate professorship.

Dr. W. D. Berg, previously visiting assistant professor at Kenyon College, has accepted a position as Senior Technical Assistant with Mutual Life Insurance Company, New York City.

Professor Emeritus H. E. Buchanan of Tulane University has been appointed Acting President of the College of the Ozarks.

Mr. Reginald Cobb, who has been a student at the University of Florida, has a position as statistician at the Naval Air Technical Training Center, Jacksonville, Florida.

Associate Professor G. M. Eddington of Tusculum College has been appointed Dean of Men.

Miss Jacqueline P. Evans has been appointed to an instructorship at Wellesley College.

Mr. Harold Glander, formerly an instructor at Superior State College, has been appointed to an instructorship at Carroll College.

Associate Professor H. G. Haefeli of Boston College has been promoted to a professorship.

Reverend C. F. Hartmann, previously associate dean of the College of Arts and Sciences of Loyola University, is now a member of the Department of Mathematics.

Acting Associate Professor Nola L. A. Haynes of the University of Missouri has been appointed to an associate professorship.

Mr. S. L. Hull of the University of Arkansas has been appointed Research Participant at the Oak Ridge National Laboratory, Tennessee.

Instructor A. R. Hyde of Trinity College, Connecticut, has accepted a position as mathematician at Emhart Manufacturing Company, Hartford, Connecticut.

Mr. T. L. Jordan, Jr., who has been a teaching fellow at Vanderbilt University, has been appointed to an associate professorship at Wofford College.

Mrs. Evelyn K. Kinney, head of the Department of Mathematics of Ferrum Junior College, Virginia, has been appointed to an associate professorship at Illinois Wesleyan University.

Mr. Blair Kinsman of St. John's College has a position as mathematician at the Chesapeake Bay Institute, Johns Hopkins University.

Mr. D. H. Kraft, formerly a student at Ohio State University, has a position as staff mathematician at Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. S. B. Kramer, who has been a student at the Polytechnic Institute of Brooklyn, has been appointed Research Assistant in the Department of Aeronautical Engineering and Applied Mechanics of the Institute.

Assistant Professor L. L. Layton of James Millikin University has accepted a position as mathematician at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Adjunct Professor T. H. Lee of the University of South Carolina is on leave of absence for the academic year 1951–52 and is engaged in graduate work at the University of North Carolina.

Mr. G. R. Lehner, formerly a student at Loyola University, has been appointed to a graduate assistantship at the University of Wisconsin.

Dr. J. A. Lewis, who has been a research physicist at Corning Glass Works, has accepted a position as a member of the technical staff of Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey.

Professor J. J. Livers of Montana State College is on leave of absence for the academic year 1951–52 and is employed by Boeing Aircraft Company, Seattle, Washington, as an educational consultant.

Mr. J. L. Maddox, Jr., who has been a teaching fellow at Alabama Polytechnic Institute, is now employed as an engineer by Philco Corporation, Philadelphia, Pennsylvania.

Instructor A. W. Mall of St. Lawrence University has a position as mathematician at the Frankford Arsenal, Philadelphia, Pennsylvania.

Lieutenant Colonel J. D. Matheson of the United States Army has been appointed to an assistant professorship in the United States Military Academy.

Instructor J. E. Maxfield of the University of Oregon has a position as mathematician at the Naval Ordnance Test Station, China Lake, California.

Associate Professor J. R. Mayor of the University of Wisconsin has been promoted to a professorship.

Mr. W. J. McClintock, previously a teaching fellow at the University of Detroit, is now Superintendent of Schools, Trenary, Michigan.

Graduate Assistant D. C. McCune of Purdue University has been appointed Instructor in Mathematics and Research Associate in the Statistical Laboratory of the University.

Mr. J. R. McPherson, previously a student at the University of North Dakota, is teaching now at Benson County Agricultural School, Maddock. North Dakota.

Associate Professor E. J. Mickle of Ohio State University has been promoted to a professorship.

Dr. G. W. Morgan has been promoted to an assistant professorship in the Graduate Division of Applied Mathematics of Brown University.

Mr. N. G. Myers, Jr., formerly a student at Gannon College, has accepted a position as statistician at Lord Manufacturing Company, Erie, Pennsylvania.

Mr. M. V. Neisius, formerly an engineer with Firestone Tire and Rubber Company, Atlanta, Georgia, has been appointed to an assistant professorship at Georgia Institute of Technology.

Miss Leola P. Odland has been appointed to a graduate assistantship at the University of Kansas.

Associate Professor E. G. Olds of Carnegie Institute of Technology has been promoted to a professorship.

Assistant Professor T. G. Ostrom of Montana State University has been promoted to an associate professorship.

Mr. W. D. Paxton, Jr., previously a student at Pomona College, has a position as insurance inspector with Retail Credit Company, San Bernardino, California.

Mrs. Patricia C. Pearson has been appointed to a teaching fellowship at Oregon State College.

Mr. G. M. Petersen, who has been a graduate student at the University of Toronto, has been appointed Lecturer at Brandon College, Manitoba, Canada.

Mr. V. J. Rathbun, formerly a graduate student at Catholic University, has been appointed to an instructorship at Mount St. John Normal, Dayton, Ohio.

Graduate Assistant Conrad Renneman, Jr., of the University of Nebraska is now Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Langley Field, Virginia.

Mr. Gordon Ritchie is teaching at Sharbot Lake High School, Ontario, Canada.

Associate Professor P. C. Rosenbloom of Syracuse University has been appointed to an associate professorship at the University of Minnesota.

Mr. H. L. Royden, formerly a graduate student at Harvard University, has been appointed to an assistant professorship at Stanford University.

Dr. Abraham Schwartz of City College of New York has been promoted to an assistant professorship.

Associate Professor Nathan Schwid of the University of Wyoming has been promoted to a professorship.

Associate Professor C. L. Seebeck, Jr., of the University of Alabama has been promoted to a professorship.

Assistant Professor R. E. Smith of Duquesne University has been promoted to an associate professorship.

Professor R. G. Sturm of the School of Civil Engineering of Purdue University has been appointed Director of Auburn Research Foundation, Alabama Polytechnic Institute.

Mr. Henry Tucker has been appointed to an assistant professorship at Kansas State College.

Mr. J. H. Wahab, formerly a teaching fellow at the University of North Carolina, has been appointed to an assistant professorship at Georgia Institute

of Technology.

Mr. J. R. Wesson, formerly a graduate fellow at Vanderbilt University, has been appointed to an assistant professorship at Birmingham-Southern College.

Mr. E. H. Weiss, formerly a mathematician and senior programmer at the Naval Proving Ground, has received an appointment as mathematical statistician at the Bureau of Census.

Mr. Dale Woods of North Dakota Agricultural College has been appointed to an instructorship at Texas Western College.

Chancellor R. L. Flowers of Duke University died on August 24, 1951; Dr. Flowers had served the University as Professor of Mathematics for forty-three years.

Assistant Professor Emeritus F. H. Hodge of Purdue University died on August 21, 1951. He was a charter member of the Association.

Assistant Professor Emeritus Ina E. Holyroyd of Kansas State College died on October 1, 1951.

Professor Emeritus Richard Morris of Rutgers University died on September 19, 1951. He was a charter member of the Association.

Professor A. V. Richardson of Bishop's University, Lennoxville, Quebec, died on June 25, 1951.

Professor C. E. Van Horn, head of the Department of Mathematics of Wartburg College, died on September 14, 1951. He had been a member of the Association for fifteen years.

Reverend Joseph Wilczewski, professor of mathematics at Marquette University, died on March 6, 1951.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE TWELFTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twelfth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 22, 1952. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received a degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team

or individual contestants may secure an application blank from Professor L. E. Bush, 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1952. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200, and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the eleven previous competitions and examinations will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, August-September, 1947, December, 1948, August-September, 1949, 1950, and 1951.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-seven persons have been elected to membership on applications duly certified.

L. R. ABRAMSON, Student, Columbia University, New York, N. Y.

M. H. AHRENDT, M.A. (Wichita) Executive Secretary, National Council of Teachers of Mathematics, Washington, D. C.

DAVID ALBERT, B.S. (Connecticut) Production Supervisor, Talon, Inc., Hamden, Conn.

S. A. ANDERSON, M.A. (George Peabody) Asso. Professor, Memphis State College, Tenn.

LEON BANKOFF, D.D.S. (Southern California) 5410 Wilshire Blvd., Los Angeles, Calif.

F. L. BARTMAN, M.A. (Michigan) Research Engineer, Willow Run Research Center, Ypsilanti, Mich.

P. O. BELL, Ph.D. (California) Professor, University of Kansas, Lawrence, Kan.

J. M. BOSWELL, M.A. (Kentucky) President, Cumberland College, Williamsburg, Ky.

G. U. BRAUER, M.A. (Michigan) Grad. Student, University of Michigan, Ann Arbor, Mich.

G. L. BULLIS, M.A. (Wisconsin) State Teachers College, Platteville, Wis.

MRS. MARY L. CANTWELL, M.A. (Columbia) Teacher, St. Petersburg Junior College, Fla.

P. F. CAUFFMAN, M.A. (Lehigh) Head of Mathematics Department, Maryland State Teachers College, Salisbury, Md.

ABRAHAM CHARNES, Ph.D. (Illinois) Asso. Professor, Carnegie Institute of Technology, Pittsburgh, Pa.

HOWARD CHITWOOD, Student, Carson-Newman College, Jefferson City, Tenn.

P. O. CIOFFI, M.S. (New Mexico S. C.) Research Engineer, Massachusetts Institute of Technology, Cambridge, Mass.

R. M. CONKLING, M.S. (Bucknell) Grad. Student, University of Florida, Gainesville, Fla.

J. G. COX, M.S. (Alabama Poly.) Instr., Alabama Polytechnic Institute, Auburn, Ala.

C. E. DIESEN, M.A. (Minnesota) Dynamics Engineer, Bell Aircraft Corporation, Niagara Falls, N. Y.

F. F. DORSEY, L.L.B. (Boston U.) Retired Patent Attorney. 205 Garfield Place, South Orange, N. J.

MARY T. DUNLEAVY, A.M. (Columbia) Statistician, E. I. duPont de Nemours and Company, Wilmington, Del.

J. D. ESARY, A.B. (Whitman) Faculty, Chemical Corps School, Ft. McClellan, Ala.

D. T. FINKBEINER, II, Ph.D. (C.I.T.) Asso. Professor, Kenyon College, Gambier, Ohio

A. L. FISCH, M.S. (Wisconsin) Acting Instr., Ohio University, Athens, Ohio

H. D. FRIEDMAN, A.M. (Columbia) Grad. Student, Pennsylvania State College, Pa.

K. G. GETMAN, A.B. (Alfred) Assoc. Editor, John Wiley & Sons, New York, N. Y.

J. A. GIORDMAINE, Student, University of Toronto, Ontario

R. L. GLASS, M.Ed. (Peabody) Head of Mathematics Department, Texarkana College, Tex.

BETTY GROSSMAN, M.A. (Michigan) Teacher, Detroit Public Schools, Mich.

J. I. HARRIS, Student, University of Michigan, Ann Arbor, Mich.

C. L. HASSELL, JR., M.S. (Purdue) Staff Member, Sandia Corporation, Albuquerque, N. M.

MARY C. HEATH, M.S. (Kentucky) Instr., Florida State University, Tallahassee, Fla.

E. E. HECKROTH, B.A. (Buffalo) Computer, Bell Aircraft Corporation, Niagara Falls, N. Y.

E. L. HITCHCOCK, B.S. (Middle Tennessee) Compounder, Goodyear Tire and Rubber Company, Akron, Ohio

T. R. HORTON, M.S. (Florida) Asst. Commandant and Instr., Bolles School, Jacksonville, Fla.

MIRIAM O. HUGGINS, A.B. (Women's C., North Carolina) Teacher, Grainger High School, Kinston, N. C.

N. W. JOHNSON, JR., Student, Carleton College, Northfield, Minn.

W. H. JOHNSON, M.S. (N.Y.U.) Mathematician, International Business Machines Corporation, New York, N. Y.

J. E. KIMBER, JR., Student, Massachusetts Institute of Technology, Cambridge, Mass.

LEONARD KLAFTER, A.B. (George Washington) Mathematician, Army Map Service, Washington, D. C.

I. I. KOLODNER, Ph.D. (N.Y.U.) Research Associate, New York University, N. Y.

M. D. KRUSKAL, M.S. (N.Y.U.) Research Associate, Princeton University, N. J.

M. R. LAUDANTE, Student, Polytechnic Institute of Brooklyn, N. Y.

B. S. LEVINE, M.A. (Washington S. C.) Grad. Student, Columbia University, New York, N. Y.

DAVID LOEV, Student, University of Pennsylvania, Philadelphia, Pa.

A. P. LOOMER, M.A. (Acadia, Canada) Professor, Milton College, Wis.

G. R. MACLANE, Ph.D. (Rice) Asst. Professor, Rice Institute, Houston, Tex.

R. A. MAGDA, Student, University of Detroit, Mich.

D. M. MANDELBAUM, Student, Hillside High School, N. J.

G. J. MANN, M.A. (Loyola, Chicago) Asst. Professor, U. S. Naval Academy, Annapolis, Md.

J. F. MANOGUE, Student, Catholic University, Washington, D. C.

JON MATHEWS, Student, Pomona College, Claremont, Calif.

J. C. MCPHERSON, B.S. (Alabama Poly.) Assistant, Duke University, Durham, N. C.

H. J. MILLER, B.S. (Seton Hall U.) Teacher, St. Patricks High School, Elizabeth, N. J.

B. B. MISRA, M.S. (Michigan) Asst. Professor, Ravenshaw College, Cuttack, India

C. T. MOLLOY, Ph.D. (N.Y.U.) Head of Mathematics Section, Kellex Corporation, New York, N. Y.

P. M. MOSKOWITZ, B.S. (Columbia) Physicist, New York Naval Shipyard, Brooklyn, N. Y.

E. O. NELSON, M.A. (Minnesota) Instr., University of North Dakota, Grand Forks, N. D.

W. J. NEWMAN, B.S. (De Paul) Grad. Student, De Paul University, Chicago, Ill.

R. V. NIEDRAUER, Student, University of Buffalo, N. Y.

O. L. PERKINS, M.A. (Atlanta) Instr., Campbell Street High School, Daytona Beach, Fla.

W. J. PERVIN, Student, University of Michigan, Ann Arbor, Mich.

K. F. POWELL, B.S. (Pittsburgh) Customer Engineer, International Business Machines Corporation, New York, N. Y.

P. C. ROGERS, M.A. (Boston U.) Grad. Assistant, University of Maryland, College Park, Md.

BERTHOLD SCHWEIZER, B.S. (M.I.T.) Mathematician, Naval Ordnance Laboratory, White Oak, Md.

RAYMOND SEDNEY, D.Sc. (Carnegie Tech.) Aerodynamicist, Douglas Aircraft Company, Santa Monica, Calif.

H. F. SIEBER, Major, Signal Corps, Ft. Monmouth, N. J.

JEROME SIMONEL, B.A. (Paris) Justice, French Oceania High Court, Papeete, Tahiti Island, French Oceania

SISTER MARY MICHAIL, M.S. (Marquette) Chairman of Mathematics Department, Clarke College, Dubuque, Iowa

T. T. TANIMOTO, Ph.D. (Pittsburgh) Asst. Professor, Allegheny College, Meadville, Pa.

R. P. TAPSCOTT, Student, University of Kentucky, Lexington, Ky.

A. W. THOMAS, JR., Student, George Pepperdine College, Los Angeles, Calif.

R. H. THOMPSON, M.A. (Kansas) Professor, Sterling College, Kans.

R. F. TIDD, M.A. (Buffalo) Instr., Canisius College, Buffalo, N. Y.

J. E. WARD, M.S. (Northwestern) Instr., University of Wisconsin, Kenosha Extension Center, Wis.

G. P. WILLIAMS, A.B. (Columbia) Grad. Student, University of Cincinnati, Ohio

C. E. WOOD, M. A. (Columbia) Teacher, North Syracuse High School, N. Y.

E. W. WRIGHT, Student, University of Idaho, Moscow, Idaho

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-eighth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Mississippi State College, State College, Mississippi, on February 16 and 17, 1951. Professor H. F. Schroeder, Chairman of the Section, presided at the Friday afternoon session and Professor S. B. Murray, Mississippi Vice-Chairman, presided at the Saturday morning session.

There were eighty persons presented including the following thirty-six members of the Association: T. A. Bickerstaff, G. J. Corley, Mamie M. Davis, W. L. Duren, Jr., Virginia I. Felder, L. M. Garrison, M. E. Gillis, F. Monica Goen, A. C. Grimes, J. A. Hardin, R. H. Hopkins, A. S. Householder, L. H. Kanter, H. T. Karnes, C. G. Killen, R. J. Koch, A. C. Maddox, J. W. Mc-

Climans, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, S. B. Murray, Arthur Ollivier, R. L. O'Quinn, P. K. Rees, T. L. Reynolds, F. A. Rickey, H. F. Schroeder, S. W. Shelton, C. D. Smith, P. K. Smith, W. H. Spragens, M. M. Temple, V. B. Temple, B. B. Townsend, A. D. Wallace.

The following officers were elected for the coming year: Chairman, Arthur Ollivier, Mississippi State College; Louisiana Vice-Chairman, C. G. Killen, Northwestern Louisiana State College; Mississippi Vice-Chairman, Virginia I. Felder, Mississippi Southern College; Secretary-Treasurer, F. A. Rickey, Louisiana State University.

The time and place of the next annual meeting of the Section were set as February 15 and 16, 1952 at Northwestern Louisiana State College, Natchitoches, Louisiana.

The invited speaker for the meeting was Dr. A. S. Householder, Oak Ridge National Laboratory. His subject for an address at the Friday evening dinner held jointly with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was *Mathematics at Oak Ridge*. At the Saturday morning session, Dr. Householder spoke on *Nervous Systems and Computing Machines*.

The following short papers were presented:

1. *A note on certain integral products*, by Professor R. A. Miller, University of Mississippi.

In this note certain relations are determined which are satisfied by two functions of a single variable such that the product of the indefinite integrals of the two functions should equal a non-zero constant multiplied by the indefinite integral of the product of the functions. The relations are such that if one of the functions is assumed, it is possible in certain cases to determine the second function. Several different cases are treated, and as a by-product, the primitive function of certain complicated integrands is determined with greater facility than by the usual methods.

2. *The induced homomorphism theorem and some of its applications to elementary group theory*, by Professor A. D. Wallace, Tulane University.

In this note it is shown how the Induced Homomorphism Theorem enables one to give simple and direct proofs of some of the isomorphism theorems commonly considered in elementary modern algebra. It is also noted that the Induced Homomorphism Theorem applies to rings. One can thus formulate analogs, for the ring structure, of the isomorphism theorems for groups.

3. *Cycloids and roses*, by Professor V. B. Temple, Louisiana College.

It is shown that, for every n-arched hypocycloid, there is an n-petaled inscribed rose; and for every n-arched epicycloid, there is an n-petaled circumscribed rose. The cycloids and their accompanying roses are defined as pedal curves. (A pedal curve is the locus of the vertex of a right angle as it moves according to given conditions). Equations for these curves are given in generalized form along with drawings showing the special cases for $n=3$ and $n=4$.

4(a). *Geometry of the circulant*, by Mr. R. A. Stokes, student, University of Mississippi, introduced by Professor B. E. Mitchell.

A circulant is a determinant in which any row is obtained from the preceding one by passing the last element over the others into the first position. When geometric meaning is given these elements, the circulant can be interpreted geometrically. The power of a point with respect to a circle with a real trace may be expressed as a circulant of the second order. There is a one-param-

eter family of conics whose discriminant is a circulant and which has the following properties: (1) The family contains one circle (imaginary); (2) The family contains two parabolas (one proper and one degenerate); (3) The family contains one equilateral hyperbola; (4) Each member of the family is tangent to every member at points (ω, ω^3) and (ω^2, ω); there are no other points of intersection; (5) The center of every member lies on the line $X = Y$.

(b) *Brachistocrone of ascent*, by Mr. Jerry McCall, student, University of Mississippi, introduced by Professor B. E. Mitchell.

Given a plane with constant rotation about a vertical axis in the plane, the impressed field of force composed of a horizontal force $\omega^2 r$ and vertical force g acts on every particle in the plane. The path of free motion of any particle is the square of an inverse hyperbolic function. Introducing a parabolic constraining curve, there is no motion of the particle. The orthogonal trajectory of this curve is logarithmic and is a brachistochrone.

5. *Approximation by Walsh-Rademacher functions*, by Professor W. H. Spragens, University of Mississippi.

The Walsh-Rademacher system of orthogonal functions is of interest because of the simple structure of the functions and the use made of the binary number notation. This paper is a brief exposition of the development of a given function in terms of this system and contains illustrations of the nature of the "step-function" approximations afforded by the partial sums.

6. *On free vectors*, by Professor W. L. Duren, Jr., Tulane University.

As is well known, a two-dimensional vector space X over the real numbers can be represented by taking as vectors the directed line segments from the origin to each point of the plane. Another representation of the same space can be constructed by taking as vectors the so-called "free vectors" which consist of all directed line segments in the plane with a convention of equivalence. This is often done incorrectly in text books, however. The purpose of this note is to present the free vector space concept from the viewpoint of abstract algebra. It is the quotient space of the cartesian product space $X \times X$ modulo the diagonal in $X \times X$, the diagonal being a scalar product ideal. This quotient space is isomorphic with X.

7. *On a separation theorem for the zeros of the ultraspherical polynomials*, by Professor L. H. Kanter, Mississippi State College.

Let $R_n(x, \lambda) = \partial/\partial x(P_n(x, \lambda))$, where $P_n(x, \lambda)$ is the ultraspherical polynomial of degree n. If the identity $P_n(x_\nu, \lambda) \equiv 0$, where x_ν is the νth zero of $P_n(x, \lambda)$, be differentiated with respect to λ, the resultant equation may be used to determine separation theorems involving the zeros of $R_n(x, \lambda)$ and those of the corresponding ultraspherical polynomial $P_n(x, \lambda)$. Generalizations to the polynomials $R_n(x, \lambda) + c(\lambda)P_n(x, \lambda)$, where $c(\lambda)$ is an arbitrary constant depending on λ, follow without difficulty. Also it can be shown that $R_n(x, \lambda)$ may be expressed as $2\sum_{\nu=0}^{n} P_\nu(x, \lambda)T_{n-\nu}(x, \lambda)/(n-\nu)$, where $T_0/0$ is defined to be equal to 0, and T_ν is the Tchebychef polynomial of degree ν.

8. *Shadow loci*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

This paper treats of procedures in finding the equations' curves followed by a shadow. The top of a vertical pole of variable length is allowed to generate a space curve. From the top of a fixed vertical pole a light is allowed to cast the shadow of the top of the variable pole upon the horizontal plane. Several systems of algebraic and differential equations are used in studying the relationship between the equations of the space curve and the shadow curve.

9. *Some statistical applications of sequential analysis*, by C. D. Smith, University of Alabama, by title.

In this paper are considered: the probability density function; the nature of the p_1, α and p_2, β relations; the functions d_1, d, and d_2 in the random sampling process. An illustrative case from auditing practice is included.

10. *The testing program and standing committee report*, by H. T. Karnes, Louisiana State University.

This is the third of a series of annual reports of the testing program conducted by the departments of mathematics of colleges of Louisiana and Mississippi to entering freshmen. In order to accumulate more data, it was decided to continue the program indefinitely. The results of the program and work of the committee have already been used to advantage in getting the certification requirements for teachers of high school mathematics increased. The first annual Mathematics Institute was favorably reported on and the second one announced.

F. A. RICKEY, *Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held in Dallas, Texas, April 27–28, 1951. Southern Methodist University served as sponsor.

About eighty persons attended the sessions of the Association including the following forty-seven members of the Association: J. C. Bradford, Ina M. Bramblett, H. E. Bray, L. A. Colquitt, J. V. Cooke, Don Cude, Terrell Ellis, B. T. Goldbeck, L. D. Gregory, Blanche B. Grover, E. H. Hanson, E. A. Hazlewood, E. R. Heineman, J. J. Henrick, C. M. Howard, C. E. Kelley, Ruth Kissel, W. M. Koger, Ruth O. Lane, W. I. Layton, A. S. Lee, J. W. Lindsay, G. R. MacLane, Hazel L. Mason, Dorothy McCoy, W. K. McNabb, Harlan C. Miller, Leo Moser, E. D. Mouzon, C. A. Murray, T. D. Oxley, H. C. Parrish, C. J. Pipes, L. W. Ramsey, Dorothy L. Rees, C. A. Rogers, C. R. Sherer, D. P. Shore, D. W. Starr, Jennie L. Tate, F. E. Ulrich, R. S. Underwood, H. S. Vandiver, Margaret M. Welch, Mabel Williams, F. L. Wrenn, C. B. Wright.

At the business meeting the following officers were elected for the next academic year: Chairman, D. W. Starr, Southern Methodist University; Vice-Chairman, C. B. Wright, East Texas State Teachers College; Secretary-Treasurer, C. R. Sherer, Texas Christian University.

At a banquet given Friday night in the Student Union Dining Hall, H. M. Hosford, Vice-President of Southern Methodist University, was the principal speaker.

The first six papers were given Friday afternoon and the remaining papers were given Saturday morning.

1. *Surfaces*, by Professor G. R. MacLane, The Rice Institute.

2. *On the equation* $\phi(n)=\pi(n)$, by Dr. Leo Moser, Texas Technological College.

As usual, let $\phi(n)$ denote Euler's totient function, and $\pi(n)$ the number of primes not exceeding n. Using elementary methods only, the equation $\phi(n)=\pi(n)$ was shown to have exactly eight solutions, namely $n=2, 3, 4, 8, 10, 14, 20, 90$. The inequality $\phi(n)<\pi(n)$ was shown to have seven solutions. A number of related partially solved, and unsolved problems were discussed.

3. *Theorem concerning exact differential equations*, by Professor R. S. Underwood, Texas Technological College.

Most texts on differential equations solve the exact equation $Mdx+Ndy=0$ by a method which is cumbersome in itself and does not generalize easily to the case of n variables. But if these texts would borrow a little from advanced calculus and use the simple idea of integrating an exact differential expression along a broken line path, the solution would be simplified materially whether there be two or n variables.

4. *A representation of a certain class of Riemann surfaces*, by Mr. J. D. Rice, The Rice Institute, introduced by the Secretary.

5. *A theory concerning point sets*, by Professor C. J. Pipes, Southern Methodist University.

Define an abstract set property by taking as postulates certain characteristics common to the classes of sets of measure zero and sets of the first category. Then, certain results that appear as dual results for sets of measure zero and sets of the first category are found to hold true likewise for any class of sets satisfying the postulates for the abstract set property. Two particular results are discussed.

6. *The use of complex elements in metric geometry*, by Professor F. E. Ulrich, The Rice Institute.

7. *Preparation and certification of teachers in Texas*, by Mr. Lee Wilburn, Assistant Commissioner for Instructors, T. E. A., introduced by the Secretary.

8. *Relative values in the teaching of mathematics*, by Professor F. L. Wren, George Peabody College for Teachers (By invitation).

C. R. SHERER, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The nineteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Carroll College, Waukesha, Wisconsin, on Saturday, May 12, 1951, in conjunction with the Wisconsin Mathematics Council. Professor E. G. Harrell presided at the morning and afternoon sessions.

About fifty persons were present, including the following thirty-one members of the Association: K. J. Arnold, R. H. Bardell, L. J. Berner, Leonard Bristow, R. H. Bruck, H. H. Conwell, H. P. Evans, J. V. Finch, L. E. Fuller, E. G. Harrell, R. C. Huffer, J. F. Kenney, Morris Marden, A. E. May, P. E. Meadows, Genevieve T. Meyer, Marianne S. Otto, O. E. Overn, H. P. Pettit, A. C. Schaeffer, M. E. Shanks, Sister M. Mechtildis, Sister Mary Felice, Sister Mary Olive, Sister Mary Petronia, Abraham Spitzbart, J. V. Talacko, K. R. Thompson, R. D. Wagner, N. M. Watermolen, Louise A. Wolf.

At the business meeting held after the morning program, the following officers were elected for the coming year: Chairman, P. E. Meadows, Carroll College; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee Chairman, R. H. Bing, University of Wisconsin.

The program consisted of the following papers:

1. *Induction*, by Professor A. C. Schaeffer, University of Wisconsin.

Induction, not as mathematical induction, but with the meaning customary in other sciences, was the topic of this paper. Induction in this sense is a useful device for discovering what may be true in mathematics. An example was given.

2. *Epistemology and manifolds*, by Professor M. E. Shanks, Purdue University.

The author discussed various postulates for physical space which are plausible on heuristic grounds from our immediate sense experience. After some simplification, for reasons of mathematical expediency, the following conjecture was formulated. Is a separable metric space which is connected, locally connected, locally compact, n-dimensional and homogeneous a manifold? The literature pertaining to the problem was reviewed and a possible attack outlined. This attack examines the group of all homeomorphisms of the space onto itself, and a set of conjectures concerning this group was formulated which would facilitate solution of the manifold problem.

3. *The Mathematical Association of America*, by Professor R. H. Bardell, University of Wisconsin in Milwaukee.

4. *Finite hemi-groups*, by Professor L. V. Toralballa, Marquette University, introduced by Professor H. P. Pettit.

A left hemigroup Σ is a system with one composition operation, X, satisfying the following: (1) For every two elements a and b of Σ there exists uniquely aXb in Σ; (2) Σ is associative with respect to X; (3) For every element a of Σ there exists an element of a^{-1} of Σ such that for every element b of Σ, $(a^{-1}Xa)Xb=b$. Properties: The equation $aX=b$ is always uniquely solvable. Two elements a and b are homologous if there exists an element c such that $ac=a$ and $bc=b$. Homology is an equivalent relation. The equation $Xa=b$ is solvable only if a and b are homologous, in which case there is one solution in each homology class. Each homology class is a group. Two elements a and b are congruent if the inverse of a in the class of b is also the inverse of b in this class. Congruence is an equivalence relation.

5. *The Wisconsin Mathematics Council*, by Miss Margaret Striegl, Wauwatosa High School, introduced by the Secretary.

The Wisconsin Mathematics Council is affiliated with the National Council of Teachers of Mathematics and was represented by Miss Margaret A. Striegl at the annual delegate assembly at Pittsburgh in April. The delegates of fifty-two groups through their discussions and recommendations to the Board of Directors are striving to coordinate and strengthen the work of national, state and local organizations. The Wisconsin Council of about two hundred and eighty members is a group organized to encourage an active interest in mathematics and to improve the teaching of mathematics at all levels. The group meets the first Thursday in November during the annual convention of the Wisconsin Education Association. The executive committee, consisting of the six officers of the council, the past president, the presidents of the mathematics sections of the state divisional associations and area mathematics clubs, and the state representative of the NCTM, meets at various times to transact business. The research committee, representing all levels of instruction, has numerous meetings to discuss the results of investigations in mathematics areas and to prepare teaching materials and aids to publish in the *Wisconsin Teacher of Mathematics* which is issued three times a year.

6. *Progress of mathematical curriculum changes in the Milwaukee schools*, by Mr. Howard M. Aker, Assistant for Research and Educational Studies, Milwaukee, introduced by the Chairman.

A background of mathematics teaching practices in the Milwaukee public schools was presented. Progress toward a common understanding of the need for change was outlined leading to the present efforts to maintain a two-track system of mathematics in the secondary schools and a workable sequence of topics in elementary classes. Improvements in guidance procedures affecting placement of students and enlargement of testing programs were presented and the need for a supervisor in the field of mathematics to coordinate efforts toward "a continuous, sequential program of mathematics from kindergarten through grade 12 for all students" and "an adequate maintenance and remedial program" was pointed up.

7. *Mathematics education in England*, by Mr. Trevor Evans, The University of Manchester, England, and the University of Wisconsin, introduced by Professor H. P. Evans.

8. *Summary report of a questionnaire on mathematics programs for the training of secondary teachers of mathematics in the Wisconsin institutions of higher learning*, by Professor E. G. Harrell, Platteville State Teachers College.

LOUISE A. WOLF, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1-2, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, Spring, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS

KENTUCKY, University of Kentucky, Lexington.

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.

MISSOURI, Lindenwood College, St. Charles, May 2, 1952.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1952.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 26, 1952.

OHIO, April 19, 1952.

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April, 1952.

UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.

WISCONSIN

Prentice-Hall
COLLEGE TEXTS

MEANINGFUL MATHEMATICS

by H. S. KALTENBORN, Chairman, Department of Mathematics, Memphis State College

The purpose of this book is to provide the mathematical part of a liberal education. Material ranging from primitive number systems to calculus is included, while the large number of problems provide adequate drill on fundamental processes. Many fascinating topics, seldom encountered in traditional courses, are treated: for example, there is an unusual and excellent chapter on "Consumer Mathematics."

384 pages 5½" x 8½" Published 1951

Basic Mathematics for General Education

by H. C. TRIMBLE, FRANK C. BOLSER, and THOMAS L. WADE, all of Florida State University

The new arrangement and treatment of topics in this book (geared to the modern theory of general education) is original with the authors. Emphasis is on mathematics as a language rather than on theory or how-to-do-it. Mathematical elegance is sometimes sacrificed to restrict discussion to issues most interesting and helpful to the freshman student.

313 pages 5½" x 8½" Published 1950

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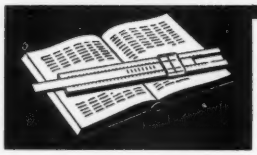

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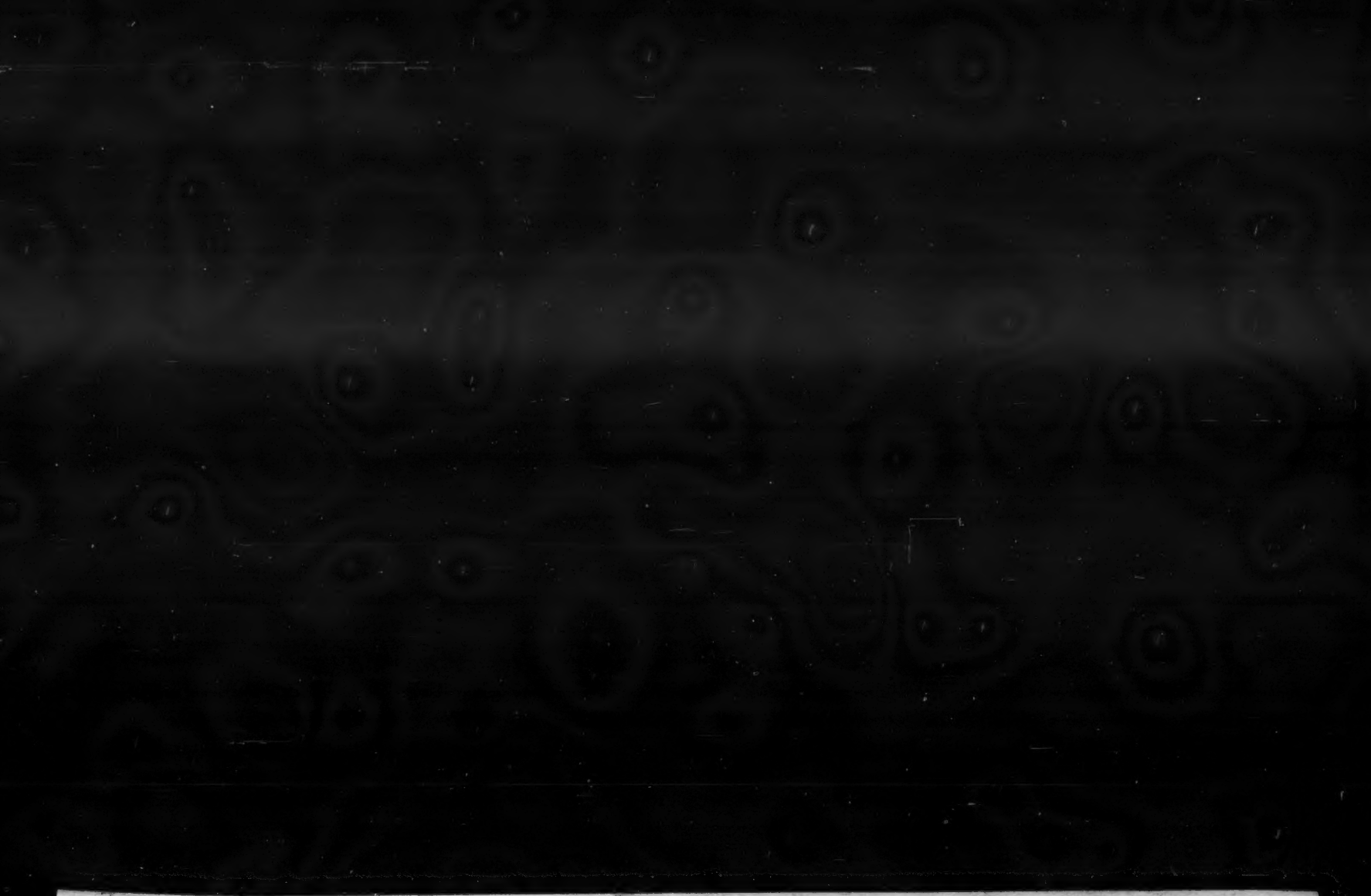

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